

6 IIR filter approximations

6.1 Introduction

This chapter deals with the design methods in which a desired frequency response is approximated by a transfer function consisting of a ratio of polynomials. In general, this type of transfer function yields an impulse response of infinite duration. Therefore, the systems approximated in this chapter are commonly referred to as infinite-duration impulse-response (IIR) filters.

In general, IIR filters are able to approximate a prescribed frequency response with fewer multiplications than FIR filters. For that matter, IIR filters can be more suitable for some practical applications, especially those ones involving real-time signal processing.

In Section 6.2, we study the classical methods of analog filter approximation, namely the Butterworth, Chebyshev, and elliptic approximations. These methods are the most widely used for approximations meeting prescribed magnitude specifications. They originated in the continuous-time domain and their use in the discrete-time domain requires an appropriate transformation.

We then address, in Section 6.3, two approaches that transform a continuous-time transfer function into a discrete-time transfer function, the impulse-invariance and bilinear transformation methods.

Section 6.4 deals with frequency transformation methods in the discrete-time domain. These methods allow the mapping of a given filter type to another, for example the transformation of a given lowpass into a desired bandpass filter.

In applications where magnitude and phase specifications are imposed, we can approximate the desired magnitude specifications by one of the classical transfer functions and design a phase equalizer to meet the phase specifications. As an alternative, we can carry out the design entirely in the digital domain, by using optimization methods to design transfer functions satisfying the magnitude and phase specifications simultaneously. Section 6.5 covers a procedure to approximate a given frequency response iteratively, employing a nonlinear optimization algorithm.

In Section 6.6, we address the situations where an IIR digital filter must present an impulse response similar to a given discrete-time sequence. This problem is commonly known as time-domain approximation.

Finally, in Section 6.7, the role of MATLAB in the approximation of IIR filters is briefly discussed.

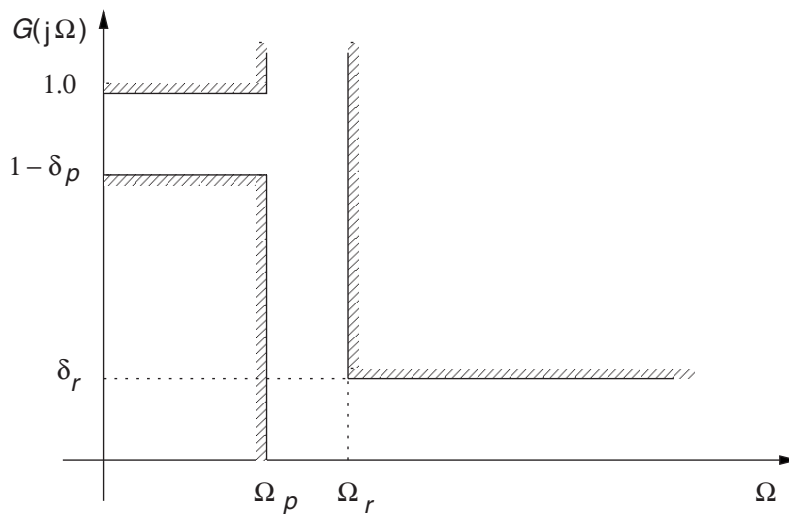


Figure 6.1 Typical gain specifications of a lowpass filter.

6.2 Analog filter approximations

This section covers the classical approximations for normalized-lowpass analog filters.¹ The other types of filters, such as the denormalized-lowpass, highpass, bandstop, and bandpass filters, are obtained from the normalized-lowpass prototype through frequency transformations, which are also addressed in this section.

6.2.1 Analog filter specification

An important step in the design of an analog filter is the definition of the desired magnitude and/or phase specifications that should be satisfied by the filter frequency response. Usually, a classical analog filter is specified through a region of the $\Omega \times H(j\Omega)$ plane² where its frequency response must be contained. This is illustrated in Figure 6.1 for a lowpass filter. In this figure, Ω_p and Ω_r denote the passband and stopband edge frequencies, respectively. The frequency region between Ω_p and Ω_r is the so-called transition band where no specification is provided. In addition, the maximum ripples in the passband and the stopband are denoted by δ_p and δ_r , respectively.

Alternatively, the specifications can be given in decibels (dB), as shown in Figure 6.2a, in the case of gain specifications. Figure 6.2b shows the same filter specified in terms of attenuation instead of gain. The relationships between the parameters of these three representations are given in Table 6.1.

¹ Normalized filters are derived from standard ones through a simple variable scaling. The original filter is then determined by reversing the frequency transformation previously applied. In this section, to avoid any source of confusion, a normalized analog frequency is always denoted by a primed variable such as Ω' .

² Note that once more Ω usually refers to analog frequency and ω to digital frequency.

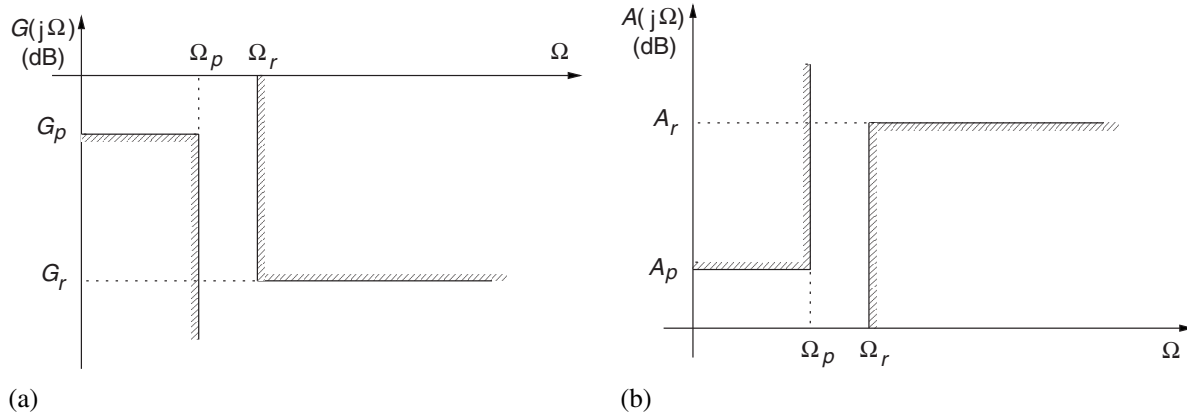


Figure 6.2 Typical specifications of a lowpass filter in dB: (a) gain; (b) attenuation.

Table 6.1. Relationships among the parameters for the gain, gain in dB, and attenuation in dB specification formats.

	Ripple	Gain [dB]	Attenuation [dB]
Passband	δ_p	$G_p = 20 \log_{10}(1 - \delta_p)$	$A_p = -G_p$
Stopband	δ_r	$G_r = 20 \log_{10} \delta_r$	$A_r = -G_r$

For historical reasons, in this chapter, we work with the attenuation specifications in dB. Using the relationships given in Table 6.1, readers should be able to transform any other format into the set of parameters that characterize the attenuation in dB.

6.2.2 Butterworth approximation

Usually, the attenuation of an all-pole normalized-lowpass filter is expressed by an equation of the following type:

$$|A(j\Omega')|^2 = 1 + |E(j\Omega')|^2 \quad (6.1)$$

where $A(s')$ is the desired attenuation function and $E(s')$ is a polynomial which has low magnitude at low frequencies and large magnitude at high frequencies.

The Butterworth approximation is characterized by a maximally flat magnitude response at $\Omega' = 0$. In order to achieve this property, we choose $E(j\Omega')$ as

$$E(j\Omega') = \epsilon (j\Omega')^n \quad (6.2)$$

where ϵ is a constant and n is the filter order. Equation (6.1) then becomes

$$|A(j\Omega')|^2 = 1 + \epsilon^2 (\Omega')^{2n} \quad (6.3)$$

resulting in the fact that the first $(2n - 1)$ derivatives of the attenuation function at $\Omega' = 0$ are equal to zero, as desired in the Butterworth approximation.

The choice of the parameter ϵ depends on the maximum attenuation A_p allowed in the passband. In that manner, since

$$A_{dB}(\Omega') = 20 \log_{10} |A(j\Omega')| = 10 \log_{10} [1 + \epsilon^2 (\Omega')^{2n}] \quad (6.4)$$

at $\Omega' = 1$, we must have that

$$A_p = A_{dB}(1) = 10 \log_{10} (1 + \epsilon^2) \quad (6.5)$$

and then

$$\epsilon = \sqrt{10^{0.1A_p} - 1} \quad (6.6)$$

To determine the filter order required to meet the attenuation specification, A_r , in the stopband, at $\Omega' = \Omega'_r$ we must have that

$$A_r = A_{dB}(\Omega'_r) = 10 \log_{10} [1 + \epsilon^2 (\Omega'_r)^{2n}] \quad (6.7)$$

Therefore, n should be smallest integer such that

$$n \geq \frac{\log_{10} \left(\frac{10^{0.1A_r} - 1}{\epsilon^2} \right)}{2 \log_{10} \Omega'_r} \quad (6.8)$$

with ϵ as in equation (6.6).

With n and ϵ available, one has to find the transfer function $A(s')$. We can factor $|A(j\Omega')|^2$ in equation (6.3) as

$$|A(j\Omega')|^2 = A(-j\Omega')A(j\Omega') = 1 + \epsilon^2 \Omega'^{2n} = 1 + \epsilon^2 [-(j\Omega')^2]^n \quad (6.9)$$

Using the analytical continuation for complex variables (Churchill, 1975), that is, replacing $j\Omega'$ by s' , we have that

$$A(s')A(-s') = 1 + \epsilon^2 (-s'^2)^n \quad (6.10)$$

In order to determine $A(s')$, we must then find the roots of $[1 + \epsilon^2 (-s'^2)^n]$ and then choose which ones belong to $A(s')$ and which ones belong to $A(-s')$. The solutions of

$$1 + \epsilon^2 (-s'^2)^n = 0 \quad (6.11)$$

are

$$s_i = \frac{1}{\epsilon^{\frac{1}{n}}} e^{j\frac{\pi}{2}(\frac{2i+n+1}{n})} \quad (6.12)$$

with $i = 1, 2, \dots, 2n$. These $2n$ roots are located at equally spaced positions on the circumference of radius $\epsilon^{\frac{1}{n}}$ centered at the origin of the s plane. In order to obtain a

stable filter, we choose the n roots p_i on the left-hand side of the s plane to belong to the polynomial $A(s')$. As a result, the normalized transfer function is obtained as

$$H'(s') = \frac{H'_0}{A(s')} = \frac{H'_0}{\prod_{i=1}^n (s' - p_i)} \quad (6.13)$$

where H'_0 is chosen so that $|H'(j0)| = 1$, and thus

$$H'_0 = \prod_{i=1}^n (-p_i) \quad (6.14)$$

An important characteristic of the Butterworth approximation is that its attenuation increases monotonically with frequency. In addition, it increases very slowly in the passband and quickly in the stopband. In the Butterworth approximation, if one wants to increase the attenuation one has to increase the filter order. However, if one sacrifices the monotonicity of the attenuation, a higher attenuation in the stopband can be obtained for the same filter order. A classic example of one such approximation is the Chebyshev approximation.

6.2.3 Chebyshev approximation

The attenuation function of a normalized-lowpass Chebyshev filter is characterized by

$$|A(j\Omega')|^2 = 1 + \epsilon^2 C_n^2(\Omega') \quad (6.15)$$

where $C_n(\Omega')$ is a Chebyshev function of order n , which can be written in its trigonometric form as

$$C_n(\Omega') = \begin{cases} \cos(n \cos^{-1} \Omega'), & 0 \leq \Omega' \leq 1 \\ \cosh(n \cosh^{-1} \Omega'), & \Omega' > 1 \end{cases} \quad (6.16)$$

These functions $C_n(\Omega')$ have the following properties

$$\begin{cases} 0 \leq C_n^2(\Omega') \leq 1, & 0 \leq \Omega' \leq 1 \\ C_n^2(\Omega') > 1, & \Omega' > 1 \end{cases} \quad (6.17)$$

As a consequence, for the attenuation function defined in equation (6.15), the passband is placed in the frequency range $0 \leq \Omega' \leq 1$, the rejection band is in the range $\Omega' > 1$, as desired, and the parameter ϵ once again determines the maximum passband ripple.

The Chebyshev functions defined above can also be expressed in polynomial form as

$$\begin{aligned} C_{n+1}(\Omega') + C_{n-1}(\Omega') &= \cos[(n+1) \cos^{-1} \Omega'] + \cos[(n-1) \cos^{-1} \Omega'] \\ &= 2 \cos(\cos^{-1} \Omega') \cos(n \cos^{-1} \Omega') \\ &= 2\Omega' C_n(\Omega') \end{aligned} \quad (6.18)$$

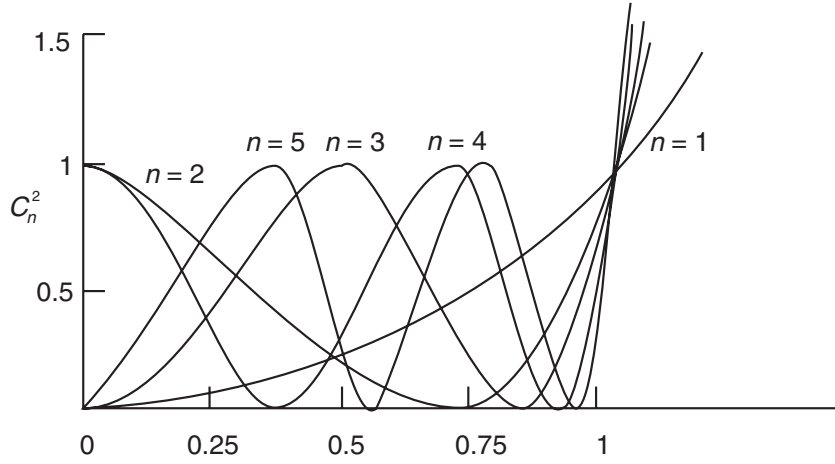


Figure 6.3 Chebyshev functions for $n = 1, 2, \dots, 5$.

with $C_0(\Omega') = 1$ and $C_1(\Omega') = \Omega'$. We can then generate higher order Chebyshev polynomials through the recursive relation above, that is

$$\begin{aligned} C_2(\Omega') &= 2\Omega'^2 - 1 \\ C_3(\Omega') &= 4\Omega'^3 - 3\Omega' \\ &\vdots \\ C_{n+1}(\Omega') &= 2\Omega' C_n(\Omega') - C_{n-1}(\Omega') \end{aligned} \quad (6.19)$$

Figure 6.3 depicts the Chebyshev functions for several values of n .

Since $C_n(\Omega') = 1$ at $\Omega' = 1$, we have that

$$A_p = A_{dB}(1) = 10 \log_{10}(1 + \epsilon^2) \quad (6.20)$$

and then

$$\epsilon = \sqrt{10^{0.1A_p} - 1} \quad (6.21)$$

From equations (6.15) and (6.16), when $\Omega' = \Omega'_r$, we find

$$A_r = A_{dB}(\Omega'_r) = 10 \log_{10} \left[1 + \epsilon^2 \cosh^2 \left(n \cosh^{-1} \Omega'_r \right) \right] \quad (6.22)$$

and thus the order of the normalized-lowpass Chebyshev filter that satisfies the required stopband attenuation is the smallest integer number that satisfies

$$n \geq \frac{\cosh^{-1} \sqrt{\frac{10^{0.1A_r} - 1}{\epsilon^2}}}{\cosh^{-1} \Omega'_r} \quad (6.23)$$

Similarly to the Butterworth case (see equation (6.9)), we can now continue the approximation process by evaluating the zeros of $A(s')A(-s')$, with $s' = j\Omega'$. Since

zero attenuation can never occur in the stopband, these zeros are in the passband region $0 \leq \Omega' \leq 1$, and thus, from equation (6.16), we have

$$\cos \left(n \cos^{-1} \frac{s'}{j} \right) = \pm \frac{j}{\epsilon} \quad (6.24)$$

The above equation can be solved for s' by defining a complex variable p as

$$p = x_1 + jx_2 = \cos^{-1} \left(\frac{s'}{j} \right) \quad (6.25)$$

Replacing this value of p in equation (6.24), we arrive at

$$\cos n(x_1 + jx_2) = (\cos nx_1 \cosh nx_2) - j(\sin nx_1 \sinh nx_2) = \pm \frac{j}{\epsilon} \quad (6.26)$$

By equating the real parts of both sides of the above equation, we can deduce that

$$\cos nx_1 \cosh nx_2 = 0 \quad (6.27)$$

and considering that

$$\cosh nx_2 \geq 1, \quad \forall n, x_2 \quad (6.28)$$

we then have

$$\cos nx_1 = 0 \quad (6.29)$$

which yields the following $2n$ solutions:

$$x_{1i} = \frac{2i + 1}{2n} \pi \quad (6.30)$$

for $i = 0, 1, \dots, (2n - 1)$. Now, by equating the imaginary parts of both sides of equation (6.26) and using the values of x_{1i} obtained in equation (6.30), it follows that

$$\sin nx_{1i} = \pm 1 \quad (6.31)$$

$$x_2 = \frac{1}{n} \sinh^{-1} \left(\frac{1}{\epsilon} \right) \quad (6.32)$$

Since, from equations (6.24) and (6.25), the zeros of $A(s')A(-s')$ are given by

$$s'_i = \sigma'_i \pm j\Omega'_i = j \cos(x_{1i} + jx_2) = \sin x_{1i} \sinh x_2 + j \cos x_{1i} \cosh x_2 \quad (6.33)$$

for $i = 0, 1, \dots, (2n - 1)$, we have, from equations (6.25) and (6.30), that

$$\sigma_i = \pm \sin \frac{\pi}{2} \left(\frac{2i + 1}{n} \right) \sinh \left(\frac{1}{n} \sinh^{-1} \frac{1}{\epsilon} \right) \quad (6.34)$$

$$\Omega_i = \cos \frac{\pi}{2} \left(\frac{2i + 1}{n} \right) \cosh \left(\frac{1}{n} \sinh^{-1} \frac{1}{\epsilon} \right) \quad (6.35)$$

The calculated zeros belong to $A(s')A(-s')$. Analogously to the Butterworth case, we associate the n zeros, p_i , with negative real part to $A(s')$, in order to guarantee the filter stability.

The above equations indicate that the zeros of a Chebyshev approximation are placed on an ellipse in the s plane, since equation (6.34) implies the following relation:

$$\left[\frac{\sigma_i}{\sinh(\frac{1}{n} \sinh^{-1} \frac{1}{\epsilon})} \right]^2 + \left[\frac{\Omega_i}{\cosh(\frac{1}{n} \sinh^{-1} \frac{1}{\epsilon})} \right]^2 = 1 \quad (6.36)$$

The transfer function of the Chebyshev filter is then given by

$$H'(s') = \frac{H'_0}{A(s')} = \frac{H'_0}{\prod_{i=1}^n (s' - p_i)} \quad (6.37)$$

where H'_0 is chosen so that $A(s')$ satisfies equation (6.15), that is

$$H'_0 = \begin{cases} \prod_{i=1}^n (-p_i), & \text{for } n \text{ odd} \\ 10^{-0.05A_p} \prod_{i=1}^n (-p_i), & \text{for } n \text{ even} \end{cases} \quad (6.38)$$

It is interesting to note that in the Butterworth case the frequency response is monotone in both passband and stopband, and is maximally flat at $\Omega = 0$. In the case of the Chebyshev filters, the smooth passband characteristics are exchanged for steeper transition bands for the same filter orders. In fact, for a given prescribed specification, Chebyshev filters usually require lower-order transfer functions than Butterworth filters, owing to their equiripple behavior in the passband.

6.2.4 Elliptic approximation

The two approximations discussed so far, namely the lowpass Butterworth and Chebyshev approximations, lead to transfer functions whose numerator is a constant and the denominator is a polynomial in s . These are called all-pole filters, because all their zeros are located at infinity. When going from the Butterworth to the Chebyshev filters, we have traded-off monotonicity and maximal flatness in the passband for higher attenuation in the stopband. At this point, it is natural to wonder whether we could also exchange the monotonicity in the stopband possessed by the Butterworth and Chebyshev filters for an even steeper transition band without increasing the filter order. This is indeed the case, as approximations with finite-frequency zeros can have transition bands with very steep slopes.

In practice, there are transfer function approximations with finite zeros which have equiripple characteristics in the passband and in the stopband, with the advantage that

their coefficients can be computed using closed formulas. These filters are usually called elliptic filters, as their closed-form equations are derived based on elliptic functions, but they are also known as Cauer or Zolotarev filters (Daniels, 1974).

This section covers the lowpass elliptic filter approximation. (The derivations are not detailed here, as they are beyond the scope of this book.) In the following, we describe an algorithm to calculate the coefficients of elliptic filters which is based on the procedure described in the benchmark book (Antoniou, 1993).

Consider the following lowpass filter transfer function:

$$|H(j\Omega')| = \frac{1}{\sqrt{1 + R_n^2(\Omega')}} \quad (6.39)$$

where

$$R_n(\Omega') = \begin{cases} k_p \frac{\prod_{i=1}^{\frac{n}{2}} (\Omega_i^2 \Omega_r - \Omega'^2)}{\prod_{i=1}^{\frac{n}{2}} (\Omega_r - \Omega_i'^2)}, & \text{for } n \text{ even} \\ k_i \Omega' \frac{\prod_{i=1}^{\frac{n-1}{2}} (\Omega_i^2 \Omega_r - \Omega'^2)}{\prod_{i=1}^{\frac{n-1}{2}} (\Omega_r - \Omega_i^2 \Omega'^2)}, & \text{for } n \text{ odd} \end{cases} \quad (6.40)$$

The computation of $R_n(\Omega')$ requires the use of some elliptic functions. The normalization procedure for the elliptic approximation is rather distinct from the one for the Butterworth and Chebyshev filters. Here, the frequency normalization factor is given by

$$\Omega_c = \sqrt{\Omega_p \Omega_r} \quad (6.41)$$

In that manner, we have that

$$\Omega'_p = \sqrt{\frac{\Omega_p}{\Omega_r}} \quad (6.42)$$

$$\Omega'_r = \sqrt{\frac{\Omega_r}{\Omega_p}} \quad (6.43)$$

Defining

$$k = \frac{\Omega'_p}{\Omega'_r} \quad (6.44)$$

$$q_0 = \frac{1}{2} \left[\frac{1 - (1 - k^2)^{\frac{1}{4}}}{1 + (1 - k^2)^{\frac{1}{4}}} \right] \quad (6.45)$$

$$q = q_0 + 2q_0^5 + 15q_0^9 + 150q_0^{13} \quad (6.46)$$

$$\epsilon = \sqrt{\frac{10^{0.1A_p} - 1}{10^{0.1A_r} - 1}} \quad (6.47)$$

the specifications are satisfied if the filter order n is chosen through the following relation:

$$n \geq \frac{\log_{10} \frac{16}{\epsilon^2}}{\log_{10} \frac{1}{q}} \quad (6.48)$$

Having the filter order n , we can then determine the following parameters before proceeding with the computation of the filter coefficients:

$$\Theta = \frac{1}{2n} \ln \frac{10^{0.05A_p} + 1}{10^{0.05A_p} - 1} \quad (6.49)$$

$$\sigma = \left| \frac{2q^{\frac{1}{4}} \sum_{j=0}^{\infty} (-1)^j q^{j(j+1)} \sinh[(2j+1)\Theta]}{1 + 2 \sum_{j=1}^{\infty} (-1)^j q^{j^2} \cosh(2j\Theta)} \right| \quad (6.50)$$

$$W = \sqrt{(1 + k\sigma^2) \left(1 + \frac{\sigma^2}{k} \right)} \quad (6.51)$$

Also, for $i = 1, 2, \dots, l$, we compute

$$\rho_i = \frac{2q^{\frac{1}{4}} \sum_{j=0}^{\infty} (-1)^j q^{j(j+1)} \sin \frac{(2j+1)\pi u}{n}}{1 + 2 \sum_{j=1}^{\infty} (-1)^j q^{j^2} \cos \frac{2j\pi u}{n}} \quad (6.52)$$

$$V_i = \sqrt{(1 - k\rho_i^2) \left(1 - \frac{\rho_i^2}{k} \right)} \quad (6.53)$$

where

$$\left. \begin{aligned} u &= i, & \text{for } n \text{ odd} \\ u &= i - \frac{1}{2}, & \text{for } n \text{ even} \end{aligned} \right\} \quad (6.54)$$

The infinite summations in equations (6.50) and (6.52) converge extremely quickly, and only two or three terms are sufficient to reach a very accurate result.

The transfer function of a normalized-lowpass elliptic filter can be written as

$$H'(s') = \frac{H'_0}{(s' + \sigma)^m} \prod_{i=1}^l \frac{s'^2 + b_{2i}}{s'^2 + a_{1i}s' + a_{2i}} \quad (6.55)$$

where

$$\left. \begin{aligned} m &= 0 \text{ and } l = \frac{n}{2}, & \text{for } n \text{ even} \\ m &= 1 \text{ and } l = \frac{n-1}{2}, & \text{for } n \text{ odd} \end{aligned} \right\} \quad (6.56)$$

The coefficients of the above transfer function are calculated based on the parameters obtained from equations (6.44)–(6.53) as

$$b_{2i} = \frac{1}{\rho_i^2} \quad (6.57)$$

$$a_{2i} = \frac{(\sigma V_i)^2 + (\rho_i W)^2}{(1 + \sigma^2 \rho_i^2)^2} \quad (6.58)$$

$$a_{1i} = \frac{2\sigma V_i}{1 + \sigma^2 \rho_i^2} \quad (6.59)$$

$$H'_0 = \begin{cases} \sigma \prod_{i=1}^l \frac{a_{2i}}{b_{2i}}, & \text{for } n \text{ odd} \\ 10^{-0.05A_p} \sigma \prod_{i=1}^l \frac{a_{2i}}{b_{2i}}, & \text{for } n \text{ even} \end{cases} \quad (6.60)$$

Following this procedure, the resulting minimum stopband attenuation is slightly better than the specified value, being precisely given by

$$A_r = 10 \log_{10} \left(\frac{10^{0.1A_p} - 1}{16q^n} + 1 \right) \quad (6.61)$$

6.2.5 Frequency transformations

The approximation methods presented so far are meant for designing normalized-lowpass filters. In this subsection, we address the issue of how a transfer function of a general lowpass, highpass, symmetric bandpass, or symmetric bandstop filter

can be transformed into a normalized-lowpass transfer function, and vice versa. The procedure used here, the so-called frequency transformation technique, consists of replacing the variable s' in the normalized-lowpass filter by an appropriate function of s . In the following, we make a detailed analysis of the normalized-lowpass $\leftrightarrow$ bandpass transformation. The analyses of the other transformations are similar, and their expressions are summarized in Table 6.2.

A normalized-lowpass transfer function $H'(s')$ can be transformed into a symmetric bandpass transfer function by applying the following substitution of variables

$$s' \leftrightarrow \frac{1}{a} \frac{s^2 + \Omega_0^2}{Bs} \quad (6.62)$$

where Ω_0 is the central frequency of the bandpass filter, B is the filter passband width, and a is a normalization parameter that depends upon the filter type, as follows:

$$\Omega_0 = \sqrt{\Omega_{p1} \Omega_{p2}} \quad (6.63)$$

$$B = \Omega_{p2} - \Omega_{p1} \quad (6.64)$$

$$a = \begin{cases} 1, & \text{for any Butterworth or Chebyshev filter} \\ \sqrt{\frac{\Omega_r}{\Omega_p}}, & \text{for a lowpass elliptic filter} \\ \sqrt{\frac{\Omega_p}{\Omega_r}}, & \text{for a highpass elliptic filter} \\ \sqrt{\frac{\Omega_{r2} - \Omega_{r1}}{\Omega_{p2} - \Omega_{p1}}}, & \text{for a bandpass elliptic filter} \\ \sqrt{\frac{\Omega_{p2} - \Omega_{p1}}{\Omega_{r2} - \Omega_{r1}}}, & \text{for a bandstop elliptic filter} \end{cases} \quad (6.65)$$

The value of a is different from unity for the elliptic filters because in this case the normalization is not $\Omega'_p = 1$, but $\sqrt{\Omega'_p \Omega'_r} = 1$ (see equations (6.41)–(6.43)).

The frequency transformation in equation (6.62) has the following properties:

- The frequency $s' = j0$ is transformed into $s = \pm j\Omega_0$.
- Any complex frequency $s' = -j\Omega'$, corresponding to an attenuation of A_{dB} in the normalized-lowpass filter, is transformed into the two distinct frequencies

$$\Omega_1 = -\frac{1}{2}aB\Omega' + \sqrt{\frac{1}{4}a^2B^2\Omega'^2 + \Omega_0^2} \quad (6.66)$$

$$\bar{\Omega}_1 = -\frac{1}{2}aB\Omega' - \sqrt{\frac{1}{4}a^2B^2\Omega'^2 + \Omega_0^2} \quad (6.67)$$

where Ω_1 is a positive frequency and $\bar{\Omega}_1$ is a negative frequency, both corresponding to the attenuation A_{dB} .

In addition, a complex frequency $s' = j\Omega'$, which also corresponds to an attenuation of A_{dB} , is transformed into two frequencies with the same attenuation level, that is,

$$\Omega_2 = \frac{1}{2}aB\Omega' + \sqrt{\frac{1}{4}a^2B^2\Omega'^2 + \Omega_0^2} \quad (6.68)$$

$$\bar{\Omega}_2 = \frac{1}{2}aB\Omega' - \sqrt{\frac{1}{4}a^2B^2\Omega'^2 + \Omega_0^2} \quad (6.69)$$

and it can be seen that $\bar{\Omega}_1 = -\Omega_2$ and $\bar{\Omega}_2 = -\Omega_1$.

The positive frequencies Ω_1 and Ω_2 are the ones we are interested in analyzing. They can be expressed in a single equation as follows:

$$\Omega_{1,2} = \mp \frac{1}{2}aB\Omega' + \sqrt{\frac{1}{4}a^2B^2\Omega'^2 + \Omega_0^2} \quad (6.70)$$

from which we get

$$\Omega_2 - \Omega_1 = aB\Omega' \quad (6.71)$$

$$\Omega_1\Omega_2 = \Omega_0^2 \quad (6.72)$$

These relationships indicate that, in this kind of transformation, for each frequency with attenuation A_{dB} there is another frequency geometrically symmetric with respect to the central frequency Ω_0 with the same attenuation.

- From the above the cutoff frequency of the normalized-lowpass filter Ω'_p is mapped into the frequencies

$$\Omega_{p1,2} = \mp \frac{1}{2}aB\Omega'_p + \sqrt{\frac{1}{4}a^2B^2\Omega_p'^2 + \Omega_0^2} \quad (6.73)$$

such that

$$\Omega_{p2} - \Omega_{p1} = aB\Omega'_p \quad (6.74)$$

$$\Omega_{p1}\Omega_{p2} = \Omega_0^2 \quad (6.75)$$

- Similarly, the stopband edge frequency Ω'_r of the normalized-lowpass prototype is transformed into the frequencies

$$\Omega_{r1,2} = \mp \frac{1}{2}aB\Omega'_r + \sqrt{\frac{1}{4}a^2B^2\Omega_r'^2 + \Omega_0^2} \quad (6.76)$$

such that

$$\Omega_{r2} - \Omega_{r1} = aB\Omega'_r \quad (6.77)$$

$$\Omega_{r1}\Omega_{r2} = \Omega_0^2 \quad (6.78)$$

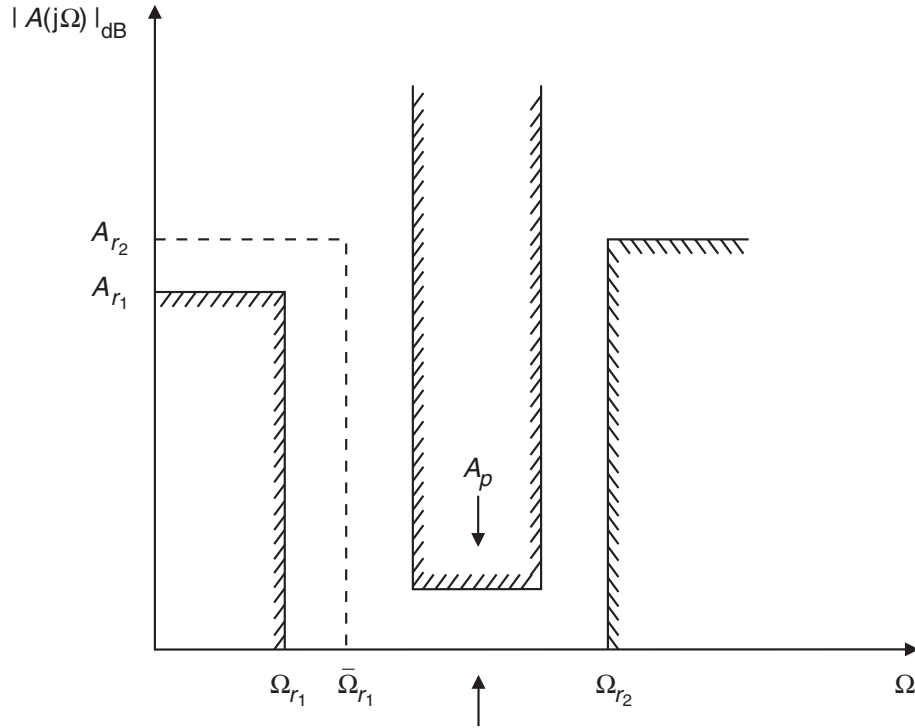


Figure 6.4 Nonsymmetric bandpass filter specifications.

The above analysis leads to the conclusion that this normalized-lowpass $\leftrightarrow$ bandpass transformation works for bandpass filters which are geometrically symmetric with respect to the central frequency. However, bandpass filter specifications are not usually geometrically symmetric in practice. Fortunately, we can generate geometrically symmetric bandpass specifications satisfying the minimum stopband attenuation requirements by the following procedure (see Figure 6.4):

- (i) Compute $\Omega_0^2 = \Omega_{p1} \Omega_{p2}$.
- (ii) Compute $\bar{\Omega}_{r1} = \frac{\Omega_0^2}{\Omega_{r2}}$, and if $\bar{\Omega}_{r1} > \Omega_{r1}$, replace Ω_{r1} with $\bar{\Omega}_{r1}$, as illustrated in Figure 6.4.
- (iii) If $\bar{\Omega}_{r1} \leq \Omega_{r1}$, then compute $\bar{\Omega}_{r2} = \frac{\Omega_0^2}{\Omega_{r1}}$, and replace Ω_{r2} with $\bar{\Omega}_{r2}$.
- (iv) If $A_{r1} \neq A_{r2}$, choose $A_r = \max\{A_{r1}, A_{r2}\}$.

Once the geometrically symmetric bandpass filter specifications are available, we need to determine the normalized frequencies Ω'_p and Ω'_r , in order to have the corresponding normalized-lowpass filter completely specified. According to equations (6.74) and (6.77), they can be computed as follows:

$$\Omega'_p = \frac{1}{a} \quad (6.79)$$

$$\Omega'_r = \frac{1}{a} \frac{\Omega_{r2} - \Omega_{r1}}{\Omega_{p2} - \Omega_{p1}} \quad (6.80)$$

Table 6.2. Analog frequency transformations.

Transformation	Normalization	Denormalization
lowpass(Ω) $\leftrightarrow$ lowpass(Ω')	$\Omega'_p = \frac{1}{a}$ $\Omega'_r = \frac{1}{a} \frac{\Omega_r}{\Omega_p}$	$s' \leftrightarrow \frac{1}{a} \frac{s}{\Omega_p}$
highpass(Ω) $\leftrightarrow$ lowpass(Ω')	$\Omega'_p = \frac{1}{a}$ $\Omega'_r = \frac{1}{a} \frac{\Omega_p}{\Omega_r}$	$s' \leftrightarrow \frac{1}{a} \frac{\Omega_p}{s}$
bandpass(Ω) $\leftrightarrow$ lowpass(Ω')	$\Omega'_p = \frac{1}{a}$ $\Omega'_r = \frac{1}{a} \frac{\Omega_{r2} - \Omega_{r1}}{\Omega_{p2} - \Omega_{p1}}$	$s' \leftrightarrow \frac{1}{a} \frac{s^2 + \Omega_0^2}{Bs}$
bandstop(Ω) $\leftrightarrow$ lowpass(Ω')	$\Omega'_p = \frac{1}{a}$ $\Omega'_r = \frac{1}{a} \frac{\Omega_{p2} - \Omega_{p1}}{\Omega_{r2} - \Omega_{r1}}$	$s' \leftrightarrow \frac{1}{a} \frac{Bs}{s^2 + \Omega_0^2}$

It is worth noting that bandstop filter specifications must also be geometrically symmetric. In this case, however, in order to satisfy the minimum stopband attenuation requirements, the stopband edges must be preserved, while the passband edges should be modified analogously to the procedure described above.

A summary of all types of transformations, including the respective correspondence among the lowpass prototype frequencies and desired filter specifications, are shown in Table 6.2.

The general procedure to approximate a standard analog filter using frequency transformations can be summarized as follows:

- (i) Determine the specifications for the lowpass, highpass, bandpass, or bandstop analog filter.
- (ii) When designing a bandpass or bandstop filter, make sure the specifications are geometrically symmetric following the proper procedure described earlier in this subsection.
- (iii) Determine the normalized-lowpass specifications equivalent to the desired filter, following the relationships seen in Table 6.2.
- (iv) Perform the filter approximation using the Butterworth, Chebyshev, or elliptic methods.
- (v) Denormalize the prototype using the frequency transformations given on the right-hand side of Table 6.2.

Sometimes the approximation of analog filters can present poor numerical conditioning, especially when the desired filter has a narrow transition and/or passband. In this case, design techniques employing transformed variables are available (Daniels, 1974; Sedra & Brackett, 1978), which can improve the numerical conditioning by separating the roots of the polynomials involved.

EXAMPLE 6.1

Design a bandpass filter satisfying the specification below using the Butterworth, Chebyshev, and elliptic approximation methods:

$$\left. \begin{aligned} A_p &= 1.0 \text{ dB} \\ A_r &= 40 \text{ dB} \\ \Omega_{r_1} &= 1394\pi \text{ rad/s} \\ \Omega_{p_1} &= 1510\pi \text{ rad/s} \\ \Omega_{p_2} &= 1570\pi \text{ rad/s} \\ \Omega_{r_2} &= 1704\pi \text{ rad/s} \end{aligned} \right\} \quad (6.81)$$

SOLUTION

Since $\Omega_{p_1}\Omega_{p_2} \neq \Omega_{r_1}\Omega_{r_2}$, the first step in the design is to determine the geometrically symmetric bandpass filter following the procedure described earlier in this subsection. In that manner, we get

$$\Omega_{r_2} = \bar{\Omega}_{r_2} = \frac{\Omega_0^2}{\Omega_{r_1}} = 1700.6456\pi \text{ rad/s} \quad (6.82)$$

Finding the corresponding lowpass specifications based on the transformations in Table 6.2, we have

$$\Omega'_p = \frac{1}{a} \quad (6.83)$$

$$\Omega'_r = \frac{1}{a} \frac{\bar{\Omega}_{r_2} - \Omega_{r_1}}{\Omega_{p_2} - \Omega_{p_1}} = \frac{1}{a} 5.1108 \quad (6.84)$$

where

$$a = \begin{cases} 1, & \text{for the Butterworth and Chebyshev filters} \\ 2.2607, & \text{for the elliptic filter} \end{cases} \quad (6.85)$$

- Butterworth approximation: from the specifications above, we can compute ϵ from equation (6.6), and, having ϵ , the minimum filter order required to satisfy the specifications from equation (6.8):

$$\epsilon = 0.5088 \quad (6.86)$$

$$n = 4 \quad (6.87)$$