Volume 64, Issue 2, 2020
Journal of Scientific Research
Institute of Science,
Banaras Hindu University, Varanasi, India.

Optimality Conditions for Set-Valued Minimax Programming Problems via Second-Order Contingent Epiderivative

Koushik Das*[1] and Chandal Nahak[2]

[1]Department of Mathematics, Taki Government College, Taki - 743429, West Bengal, India. koushikdas.maths@gmail.com*
[2]Department of Mathematics, Indian Institute of Technology Kharagpur, Kharagpur - 721302, West Bengal, India. cnahak@maths.iitkgp.ernet.in

Abstract—**In this paper, we establish second-order KKT sufficient optimality conditions of a set-valued minimax programming problem via second-order contingent epiderivative. We formulate second-order duals of Mond-Weir type, Wolfe type, and mixed type and further present corresponding duality results between the stated primal and dual problems under the assumption of second-order generalized cone convexity.**

Index Terms—**Contingent epiderivative, Convex cone, Duality, Set-valued map.**

I. Introduction

Minimax programming problems are special type of optimization problems which arise in many fields of mathematics, economics and operational research. In 1966, Bram (1966) and Danskin (1966, 1967) used the Lagrange multiplier rule to establish necessary optimality conditions of static minmax programming problems. Later, Schmitendorf (1977) proved necessary optimality conditions for existence of solutions of the minmax programming problems. Various types of duality theorems between primal and corresponding dual results are formulated by Tanimoto (1981) in the year of 1981. Necessary and sufficient optimality conditions for different types of minmax programming problems are studied by many authors like Bector and Bhatia (1985), Bector et al (1992), Chandra and Kumar (1995), Datta and Bhatia (1984), Demyanov and Malozehon (1974) and Zalmai (1985). They also studied duality results of various types in terms of differentiability of functions attached to the problems. In the year of 1999, Mehra and Bhatia (1999) established necessary optimality conditions of minmax programming problems. They also discussed duality theorems of Mond-Weir type between primal and dual problems with the help of generalized arcwise connectedness assumption. Later, Li et al (2008a,b) established necessary and sufficient optimality conditions of set-valued optimization problems via the notion of higher-order contingent derivative. They also introduced the higher-order Mond-Weir type dual for set-valued optimization problems and formulated the corresponding duality theorems under convexity assumption. Das and Nahak (2015) established the Karush-Kuhn-Tucker (KKT) sufficient optimality conditions of set-valued optimization problems via the assumptions of higher-order contingent derivative and generalized cone convexity. They also constructed duality theorems of various types between primal and corresponding dual problems. In 2017, Das and Nahak (2017c) established the KKT sufficient optimality conditions of set-valued minimax programming problems using the assumptions of contingent epiderivative and generalized cone convexity and proved the corresponding duality theorems of Wolfe, Mond-Weir, and mixed types.

In this paper, we establish the second-order KKT sufficient optimality conditions of a set-valued minimax programming problem via the notion of second-order contingent epiderivative. We also proved the second-order duality theorems of Wolfe, Mond-Weir, and mixed types under the assumption of second-order ρ-cone convexity.

This paper is organized as following manners. In Section 2, we give some definitions as well as preliminaries of the theory of set-valued optimization. We establish the second-order KKT sufficient conditions of a set-valued minimax programming problem (MP) in Section 3. Further, we develop various types of duality theorems between primal and corresponding dual problems with the help of the assumption of second-order generalized cone convexity.

II. Definitions and preliminaries

Let K be a nonempty subset of the m-dimensional Euclidean space $\mathbb{R}^m$. Then K is said to be a cone if $\lambda y \in K$, for all $y \in K$ and $\lambda \geq 0$. Furthermore, K is said to be non-trivial if $K \neq \{\mathbf{0}_{\mathbb{R}^m}\}$, proper if $K \neq \mathbb{R}^m$, pointed if $K \cap (-K) = \{\mathbf{0}_{\mathbb{R}^m}\}$, solid if $\text{int}(K) \neq \emptyset$, closed if $\overline{K} = K$,

*Corresponding Author
DOI: 10.37398/JSR.2020.640243

and convex if $K + K \subseteq K$, where $\mathbf{0}_{\mathbb{R}^m}$ is the zero element of $\mathbb{R}^m$ and int(K) and $\overline{K}$ represent the interior and closure of K, respectively.

The nonnegative orthant $\mathbb{R}^m_+$ of $\mathbb{R}^m$ is defined by

$$\mathbb{R}^m_+ = \{y = (y_1, ..., y_m) \in \mathbb{R}^m : y_i \geq 0, \forall i = 1, 2, ..., m\}.$$

Then $\mathbb{R}^m_+$ becomes a pointed closed solid convex cone and int$(\mathbb{R}^m_+) \cup \{\mathbf{0}_{\mathbb{R}^m}\}$ becomes a pointed solid convex cone in $\mathbb{R}^m$.

There are the following two notions of cone-orderings which are mainly used in $\mathbb{R}^m$ with respect to the pointed solid convex cone $\mathbb{R}^m_+$ of $\mathbb{R}^m$. For any two elements $y_1, y_2 \in \mathbb{R}^m$, we have

$$y_1 \leq y_2 \text{ if } y_2 - y_1 \in \mathbb{R}^m_+$$

and

$$y_1 < y_2 \text{ if } y_2 - y_1 \in \text{int}(\mathbb{R}^m_+).$$

We say $y_2 \geq y_1$, if $y_1 \leq y_2$ and $y_2 > y_1$, if $y_1 < y_2$.

There are two types of notions of minimality wrt. the pointed solid convex cone $\mathbb{R}^m_+$ of $\mathbb{R}^m$.

Definition 2.1: Let B be a nonempty subset of $\mathbb{R}^m$. Then

(i) $y' \in B$ is a minimal point of B if there is no $y \in B \backslash \{y'\}$, such that $y \leq y'$.

(ii) $y' \in B$ is a weakly minimal point of B if there exists no $y \in B$, such that $y < y'$.

The sets of minimal points and weakly minimal points of B are denoted by min(B) and w-min(B), respectively. We can characterize these sets as

$$\text{min}(B) = \{y' \in B : (y' - \mathbb{R}^m_+) \cap B = \{y'\}\}$$

and

$$\text{w-min}(B) = \{y' \in B : (y' - \text{int}(\mathbb{R}^m_+)) \cap B = \emptyset\}.$$

Similarly, the sets of maximal points and weakly maximal points of B can be defined and characterized.

We now present the notions of contingent cone as well as second-order contingent set in normed space.

Definition 2.2: Aubin (1981); Aubin and Frankowska (1990) Let Y be a real normed space, $\emptyset \neq B \subseteq Y$, and $y' \in \overline{B}$. The contingent cone to B at y', denoted by $T(B, y')$, can be defined as follows:

$y \in Y$ is an element $T(B, y')$ if there exist some sequences $\{\lambda_n\}$ in $\mathbb{R}$, with $\lambda_n \to 0^+$ and $\{y_n\}$ in Y, with $y_n \to y$, such that

$$y' + \lambda_n y_n \in B, \forall n \in \mathbb{N},$$

or, there exist some sequences $\{t_n\}$ in $\mathbb{R}$, with $t_n > 0$ and $\{y'_n\}$ in B, with $y'_n \to y'$, such that

$$t_n(y'_n - y') \to y.$$

If $y' \in \text{int}(B)$, then $T(B, y') = Y$.

Proposition 2.1: Aubin and Frankowska (1990) The contingent cone $T(B, y')$ is a closed cone, but not convex in general and $T(B, y') \subseteq \overline{\bigcup_{h>0} \frac{B - y'}{h}}$.

Definition 2.3: Aubin (1981); Aubin and Frankowska (1990); Cambini et al (1999) Let Y be a real normed space, $\emptyset \neq B \subseteq Y$, $y' \in \overline{B}$, and $u \in Y$. The second-order contingent set to B at y' in the direction u, denoted by $T^2(B, y', u)$, can be defined as

$y \in Y$ is an element $T^2(B, y', u)$ if there exist some sequences $\{\lambda_n\}$ in $\mathbb{R}$, with $\lambda_n \to 0^+$ and $\{y_n\}$ in Y, with $y_n \to y$, such that

$$y' + \lambda_n u + \frac{1}{2}{\lambda_n}^2 y_n \in B, \forall n \in \mathbb{N},$$

or, there exist some sequences $\{t_n\}$, $\{t'_n\}$ in $\mathbb{R}$, with $t_n, t'_n > 0$, $t_n \to \infty$, $t'_n \to \infty$, $\frac{t'_n}{t_n} \to 2$, and $\{y'_n\}$ in B, with $y'_n \to y'$, such that

$$t_n(y'_n - y') \to u \text{ and } t'_n(t_n(y'_n - y') - u) \to y.$$

Proposition 2.2: Zhu et al (2014) The second-order contingent set $T^2(B, y', u)$ is a closed set, but is not a cone in general. Even, $T^2(B, y', u)$ is not always convex but B is convex. Also, $T^2(B, y', \theta_Y) = T(T(B, y'), \theta_Y) = T(B, y')$.

Let X, Y be real normed spaces, K be a pointed solid convex cone in Y, and 2^Y be the set of all subsets of Y. Let $F : X \to 2^Y$ be a set-valued map from X to Y. Hence $F(x) \subseteq Y, \forall x \in X$. The domain, image, graph, and epigraph of F are defined by

$$\text{dom}(F) = \{x \in X : F(x) \neq \emptyset\},$$

$$F(A) = \bigcup_{x \in A} F(x), \text{ for any } A(\neq \emptyset) \subseteq X,$$

$$\text{gr}(F) = \{(x, y) \in X \times Y : y \in F(x)\},$$

and

$$\text{epi}(F) = \{(x, y) \in X \times Y : y \in F(x) + K\}.$$

Let A be a nonempty subset of X, $x' \in A$, $F : X \to 2^Y$ be a set-valued map, where $A \subseteq \text{dom}(F)$ and $y' \in F(x')$. The notion of contingent epiderivative of set-valued maps was introduced by Jahn and Rauh (1997).

Definition 2.4: Jahn and Rauh (1997) A single-valued map $D_\uparrow F(x', y') : X \to Y$ whose epigraph becomes identical with the contingent cone to the epigraph of F at (x', y'), i.e.,

$$\text{epi}(D_\uparrow F(x', y')) = T(\text{epi}(F), (x', y')),$$

is said to be the contingent epiderivative of F at (x', y').

When $f : X \to \mathbb{R}$ is a real single-valued map which is continuous at $x_0 \in X$ and also convex, then

$$D_\uparrow f(x_0, f(x_0))(u) = f'(x_0)(u), \forall u \in X,$$

where $f'(x_0)(u)$ denotes the directional derivative of f at x_0 in the direction u.

The notion of second-order contingent epiderivative of set-valued maps was introduced by Jahn et al (2005).

Definition 2.5: Jahn et al (2005) A single-valued map $D^2_\uparrow F(x', y', u, v) : X \to Y$ whose epigraph becomes identical the second-order contingent set to the epigraph of F at $(x', y') \in \text{gr}(F)$ in a direction $(u, v) \in X \times Y$, i.e.,

$$\text{epi}(D^2_\uparrow F(x', y', u, v)) = T^2(\text{epi}(F), (x', y'), (u, v)),$$

is called the second-order contingent epiderivative of F at (x', y') in the direction (u, v).

Proposition 2.3: Aubin and Frankowska (1990) Let $\emptyset \neq A \subseteq X$, $x' \in A$, $u \in X$, and $f : X \to Y$ be a single-valued map. Let f be twice continuously differentiable around x'. The second-order contingent epiderivative $D^2_{\uparrow} f(x', f(x'), u, f'(x')u)$ of f at $(x', f(x'))$ in the direction $(u, f'(x')u)$ is given by

$$D^2_{\uparrow} f(x', f(x'), u, f'(x')u)(x) = f'(x')x + \frac{1}{2} f''(x')(u, u),$$
$$x \in T^2(A, x', u).$$

We now recall the concept of cone convexity of set-valued maps, introduced by Borwein (1977) in the year of 1977.

Definition 2.6: Borwein (1977) Let A be a nonempty convex subset of a real normed space X. A set-valued map $F : X \to 2^Y$, with $A \subseteq \text{dom}(F)$, is said to be K-convex on A if $\forall x_1, x_2 \in A$ and $\lambda \in [0, 1]$,

$$\lambda F(x_1) + (1 - \lambda)F(x_2) \subseteq F(\lambda x_1 + (1 - \lambda)x_2) + K.$$

It is obvious that if a set-valued map $F : X \to 2^Y$ is K-convex on A, then epi(F) becomes a convex subset of $X \times Y$.

The cone convexity of set-valued maps can be represented via the notion of contingent epiderivative.

Lemma 2.1: Jahn and Rauh (1997) Let $F : X \to 2^Y$ be K-convex on a nonempty convex subset A of a real normed space X. Then for all $x, x' \in A$ and $y' \in F(x')$,

$$F(x) - y' \subseteq D_{\uparrow} F(x', y')(x - x') + K.$$

Definition 2.7: Let A be a nonempty subset of a real normed space X and $F : X \to 2^Y$ be a set-valued map, with $A \subseteq \text{dom}(F)$. Let $x', u \in A$, $y' \in F(x')$, and $v \in F(u) + K$. Assume that F is second-order contingent epiderivable at (x', y') in the direction $(u - x', v - y')$. Then F is called second-order K-convex at (x', y') in the direction $(u - x', v - y')$ on A if

$$F(x) - y' \subseteq D^2_{\uparrow} F(x', y', u - x', v - y')(x - x') + K, \forall x \in A.$$

Definition 2.8: A set-valued map $F : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ is said to be upper semicontinuous if $F^+(V) = \{x \in \mathbb{R}^n : F(x) \subseteq V\}$ is an open set in $\mathbb{R}^n$ for any open set V in $\mathbb{R}^m$.

Definition 2.9: Let B be a nonempty subset of $\mathbb{R}^m$. Then B is called $\mathbb{R}^m_+$-semicompact if every open cover of complements which is of the form $\{(y_i + \mathbb{R}^m_+)^c : y_i \in B, i \in I\}$ has a finite subcover.

Definition 2.10: A set-valued map $F : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ is said to be $\mathbb{R}^m_+$-semicompact-valued if $F(x)$ is $\mathbb{R}^m_+$-semicompact, $\forall x \in \text{dom}(F)$.

Corley (1987) established the existence theorem for maximization of set-valued optimization problems in which the objective functions are cone semicompact-valued and upper semicontinuous set-valued maps.

Theorem 2.1: Corley (1987) Let X, Y be real topological vector spaces, A be a nonempty compact subset of X, and $\overline{K}$ is pointed convex cone in Y. Let $F : X \to 2^Y$ be a upper semicontinuous and K-semicompact-valued set-valued map. Then the problem $\max \bigcup_{x \in A} F(x)$ has a maximal point.

For simplicity, let us assume $X = \mathbb{R}^m$, $Y = \mathbb{R}$, and $K = \mathbb{R}_+$. Let A be a nonempty subset of $\mathbb{R}^n$ and B be a nonempty compact subset of $\mathbb{R}^m$. Let $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to 2^{\mathbb{R}}$ and $G : \mathbb{R}^n \to 2^{\mathbb{R}^p}$ be two set-valued maps with

$$A \times B \subseteq \text{dom}(\Phi) \text{ and } A \subseteq \text{dom}(G).$$

We consider a set-valued minimax programming problem (MP).

$$\begin{array}{ll} \underset{x \in A}{\text{minimize}} & \max \bigcup_{y \in B} \Phi(x, y) \\ \text{subject to} & G(x) \cap (-\mathbb{R}^p_+) \neq \emptyset, \end{array} \tag{MP}$$

where the set-valued map $\Phi(x, .) : \mathbb{R}^m \to 2^{\mathbb{R}}$ is upper semicontinuous and $\mathbb{R}_+$-semicompact-valued on B, for all $x \in A$. Therefore, by Theorem 2.1, $\max \bigcup_{y \in B} \Phi(x, y)$ exists, for all $x \in A$. Since $\Phi(x, y) \subseteq \mathbb{R}$, for each $x \in A$ the problem $\max \bigcup_{y \in B} \Phi(x, y)$ has only one maximal point.

The feasible set of the problem (MP) is given by

$$S = \{x \in A : G(x) \cap (-\mathbb{R}^p_+) \neq \emptyset\}.$$

For $x \in A$, define following sets by

$$I(x) = \{j : 0 \in G_j(x), 1 \leq j \leq p\},$$
$$J(x) = \{1, 2, ..., p\} \setminus I(x),$$

and

$$B(x) = \{b \in B : \max \bigcup_{y \in B} \Phi(x, y) \in \Phi(x, b)\}.$$

Under the above assumptions, $B(x) \neq \emptyset$, for all $x \in A$.

Definition 2.11: Let $x' \in S$ and $z' = \max \bigcup_{y \in B} \Phi(x', y)$. Then (x', z') is said to be a minimizer of the problem (MP) if for all $x \in S$ and $z = \max \bigcup_{y \in B} \Phi(x, y)$,

$$z' \leq z.$$

For special case, when $\phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^p$ are single-valued maps, we have a minimax programming problem (Schmitendorf (1977)) as

$$\begin{array}{ll} \underset{x \in A}{\text{minimize}} & \max \bigcup_{y \in B} \phi(x, y) \\ \text{subject to} & g(x) \in (-\mathbb{R}^p_+), \end{array}$$

by considering $\Phi(x, y) = \{\phi(x, y)\}$ and $G(x) = \{g(x)\}$ in the problem (MP).

III. Main results

Das and Nahak (2014, 2015, 2016a,b, 2017a,b,c, 2020) introduced the notion of ρ-cone convexity for set-valued maps. They establish the KKT sufficient conditions and develop the duality results for various types of set-valued optimization problems under contingent epiderivative as well as ρ-cone convexity assumptions. For $\rho = 0$, we get the notion of cone convexity for set-valued maps introduced by Borwein (1977).

Definition 3.1: Das and Nahak (2014, 2016b) Let A be a nonempty convex subset of $\mathbb{R}^n$, $e \in \text{int}(\mathbb{R}^m_+)$ and $F : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ be a set-valued map, with $A \subseteq \text{dom}(F)$. Then F is said to be ρ-$\mathbb{R}^m_+$-convex on A with respect to e if there exists $\rho \in \mathbb{R}$ such that

$$\lambda F(x_1) + (1 - \lambda)F(x_2) \subseteq F(\lambda x_1 + (1 - \lambda)x_2) + \rho\lambda(1 - \lambda)\|x_1 - x_2\|^2 e + \mathbb{R}^m_+,$$
$$\forall x_1, x_2 \in A \text{ and } \forall \lambda \in [0, 1].$$

Das and Nahak (2016b) formulated a set-valued map which is ρ-cone convex but not cone convex. They also presented ρ-cone convexity for set-valued maps.

Theorem 3.1: Das and Nahak (2016b) Let A be a nonempty convex subset of $\mathbb{R}^n$, $e \in \operatorname{int}(\mathbb{R}^m_+)$ and $F : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ be ρ-$\mathbb{R}^m_+$-convex on A with respect to e. Let $x' \in A$ and $y' \in F(x')$. Then,

$$F(x)-y' \subseteq D_{\uparrow}F(x',y')(x-x')+\rho\|x-x'\|^2e+\mathbb{R}^m_+, \forall x \in A.$$

Das and Nahak (2015) also introduced the notion of second-order ρ-cone convexity for set-valued maps using second-order contingent epiderivative.

Definition 3.2: Das and Nahak (2015) Let A be a nonempty subset of $\mathbb{R}^n$, $e \in \operatorname{int}(\mathbb{R}^m_+)$, and $F : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ be a set-valued map, with $A \subseteq \operatorname{dom}(F)$. Let $x', u \in A$, $y' \in F(x')$, and $v \in F(u)+\mathbb{R}^m_+$. Assume that F is second-order contingent epiderivable at (x', y') in the direction $(u-x', v-y')$. Then F is said to be second-order ρ-$\mathbb{R}^m_+$-convex with respect to e at (x', y') in the direction $(u-x', v-y')$ on A if there exist $\rho \in \mathbb{R}$ such that

$$F(x)-y' \subseteq D^2_{\uparrow}F(x',y',u-x',v-y')(x-x')+\rho\|x-x'\|^2e+\mathbb{R}^m_+, \forall x \in A.$$

Remark 3.1: For $u = x'$ and $v = y'$, we have

$$F(x)-y' \subseteq D_{\uparrow}F(x',y')(x-x')+\rho\|x-x'\|^2e+\mathbb{R}^m_+, \forall x \in A.$$

In this case, we have the first order ρ-$\mathbb{R}^m_+$-convexity via contingent epiderivative.

If $\rho > 0$, then F is called strongly second-order ρ-$\mathbb{R}^m_+$-convex, if $\rho = 0$, we have the notion of second-order $\mathbb{R}^m_+$-convexity, and if $\rho < 0$, then F is called weakly second-order ρ-$\mathbb{R}^m_+$-convex.
Obviously, strongly second-order ρ-$\mathbb{R}^m_+$-convexity $\Rightarrow$ second-order $\mathbb{R}^m_+$-convexity $\Rightarrow$ weakly second-order ρ-$\mathbb{R}^m_+$-convexity. Das and Nahak (2015) developed a set-valued map $F : \mathbb{R} \to 2^{\mathbb{R}^2}$, which is second-order ρ-$\mathbb{R}^2_+$-convex for some ρ, but is not second-order $\mathbb{R}^m_+$-convex.

Remark 3.2: In case of single-valued map, Definition 3.2 coincides with the existing one. Let X, Y be real normed spaces, K be a pointed solid convex cone in Y, $e \in \operatorname{int}(K)$, $u \in X$, and $v \in Y$. Let $f : X \to Y$ be second-order continuously differentiable function at $x' \in X$. By considering $F(x) = \{f(x)\}$, from Definition 3.2 and Proposition 2.3, we can conclude that f is said to be second-order ρ-K-convex with respect to e at $(x', f(x'))$ in the direction $(u-x', v-f(x'))$ if there exists $\rho \in \mathbb{R}$ such that

$$f(x)-f(x') \in f'(x')(x-x')+\frac{1}{2}f''(x')(u-x',u-x') +\rho\|x-x'\|^2e+K, \forall x \in X,$$

where $v - f(x') = f'(x')(u-x')$.
The followings are some special cases.
When $Y = \mathbb{R}^m$, $K = \mathbb{R}^m_+$, $f = (f_1, f_2, ..., f_m)$, and $e = (1, 1, ..., 1) = \mathbf{1}_{\mathbb{R}^m}$, we have

$$f_i(x)-f_i(x') \geq f'_i(x')(x-x')+\frac{1}{2}f''_i(x')(u-x',u-x') +\rho\|x-x'\|^2, \forall x \in X \text{ and } i = 1, 2, ..., m.$$

When $Y = \mathbb{R}$, $K = \mathbb{R}_+$, and $e = 1$, we have

$$f(x)-f(x') \geq f'(x')(x-x')+\frac{1}{2}f''(x')(u-x',u-x') +\rho\|x-x'\|^2, \forall x \in X.$$

When $X = \mathbb{R}^n$, $Y = \mathbb{R}$, $K = \mathbb{R}_+$, and $e = 1$, we have

$$f(x)-f(x') \geq (x-x')^T\nabla f(x')+\frac{1}{2}(u-x')^TH(x')(u-x') +\rho\|x-x')\|^2, \forall x \in X,$$

where $\nabla f(x')$ and $H(x')$ are the gradient and Hessain matrix of f at x', respectively.

A. Second-order optimality conditions

The second-order KKT sufficient conditions of the set-valued minimax programming problem (MP) are developed under second-order ρ-cone convexity for set-valued maps.

Theorem 3.2: (Second-order sufficient optimality conditions) Let A be a nonempty convex subset of $\mathbb{R}^n$, $x' \in S$, and $z' = \max \bigcup_{y \in B} \Phi(x', y)$. Suppose that there exist a positive integer k, $z^*_i \geq 0$, $y_i \in B(x')$, $(1 \leq i \leq k)$ with $\sum_{i=1}^{k} z^*_i \neq 0$, and $w^*_j \geq 0$, $w'_j \in G_j(x') \cap (-\mathbb{R}_+)$, $(1 \leq j \leq p)$, such that

$$\sum_{i=1}^{k} z^*_i D^2_{\uparrow}\Phi(., y_i)(x', z', t-x', r-z')(x-x') + \sum_{j=1}^{p} w^*_j D^2_{\uparrow}G_j(x', w'_j, t-x', s-w'_j)(x-x') \geq 0, \forall x \in A \tag{III.1}$$

and

$$w^*_j w'_j = 0, \forall j = 1, 2, ..., p. \tag{III.2}$$

Let $\rho_i, \rho'_j \in \mathbb{R}$, for $i = 1, 2, ..., k$ and $j = 1, 2, ..., p$. Let $t \in A$, $r \in \Phi(t, y_i)+\mathbb{R}_+$, and $s \in G_j(u)+\mathbb{R}_+$. Assume that $\Phi(., y_i)$, $(1 \leq i \leq k)$, is second-order ρ_i-$\mathbb{R}_+$-convex at (x', z') in the direction $(t-x', r-z')$ and G_j, $(1 \leq j \leq p)$ is second-order ρ'_j-$\mathbb{R}_+$-convex (x', w'_j) in the direction $(t-x', s-w'_j)$, respectively, with respect to 1, on A, with

$$\sum_{i=1}^{k} z^*_i \rho_i + \sum_{j=1}^{p} w^*_j \rho'_j \geq 0, \tag{III.3}$$

then (x', z') is a minimizer of the problem (MP).

Proof: Assume that (x', z') is not a minimizer of the problem (MP).
Hence there exist $x \in S$ and $z = \max \bigcup_{y \in B} \Phi(x, y)$, such that

$$z < z'.$$

As $y_i \in B(x')$ $(1 \leq i \leq k)$, we have

$$\max \bigcup_{y \in B} \Phi(x', y) \in \Phi(x', y_i).$$

As $z' = \max \bigcup_{y \in B} \Phi(x', y)$, we have

$$z' \in \Phi(x', y_i), \forall i = 1, 2, ..., k.$$

Choose

$$z_i \in \Phi(x, y_i), \forall i = 1, 2, ..., k.$$

Again, as $z = \max \bigcup_{y \in B} \Phi(x, y)$ and $y_i \in B(x') \subseteq B$, we have

$$z_i \leq z.$$

Therefore,

$$z_i \leq z < z'.$$

Hence,

$$\sum_{i=1}^{k} z_i^* z_i < \sum_{i=1}^{k} z_i^* z'.$$

As $x \in S$, there exists

$$w_j \in G_j(x) \cap (-\mathbb{R}_+), (1 \leq j \leq p).$$

Since $w_j^* \geq 0$, $(1 \leq j \leq p)$, we have

$$w_j^* w_j \leq 0, \forall j = 1, 2, ..., p.$$

So,

$$\sum_{j=1}^{p} w_j^* w_j \leq 0.$$

As $w_j^* w_j' = 0, \forall j = 1, 2, ..., p$, we have

$$\sum_{j=1}^{p} w_j^* w_j \leq \sum_{j=1}^{p} w_j^* w_j'.$$

Hence,

$$\sum_{i=1}^{k} z_i^* z_i + \sum_{j=1}^{p} w_j^* w_j < \sum_{i=1}^{k} z_i^* z' + \sum_{j=1}^{p} w_j^* w_j'. \tag{III.4}$$

Since $\Phi(., y_i)$, $(1 \leq i \leq k)$, is second-order ρ_i-$\mathbb{R}_+$-convex at (x', z') in the direction $(t - x', r - z')$ and G_j, $(1 \leq j \leq p)$ is second-order ρ_j'-$\mathbb{R}_+$-convex (x', w_j') in the direction $(t - x', s - w_j')$, respectively, with respect to 1, on A, we have

$$\Phi(x, y_i) - z' \subseteq D_{\uparrow}^2 \Phi(., y_i)(x', z', t - x', r - z')(x - x') + \rho_i \|x - x'\|^2 + \mathbb{R}_+$$

and

$$G_j(x) - w_j' \subseteq D_{\uparrow}^2 G_j(x', w_j', t - x', s - w_j')(x - x') + \rho_j' \|x - x'\|^2 + \mathbb{R}_+.$$

Therefore,

$$z_i - z' \subseteq D_{\uparrow}^2 \Phi(., y_i)(x', z', t - x', r - z')(x - x') + \rho_i \|x - x'\|^2 + \mathbb{R}_+ \tag{III.5}$$

and

$$w_j - w_j' \subseteq D_{\uparrow}^2 G_j(x', w_j', t - x', s - w_j')(x - x') + \rho_j' \|x - x'\|^2 + \mathbb{R}_+. \tag{III.6}$$

From (III.1), (III.3), (III.5), and (III.6), we have

$$\sum_{i=1}^{k} z_i^* (z_i - z') + \sum_{j=1}^{p} w_j^* (w_j - w_j') \geq 0,$$

which contradicts (III.4).
Consequently, (x', z') is a minimizer of the problem (MP). ■

B. Second-order Mond-Weir type dual

We consider a second-order Mond-Weir type dual (MWD) of the problem (MP), where $\Phi(., y_i)$ and G_j are second-order contingent epiderivable set-valued maps, where $y_i \in B(x')$ and $x' \in A$.

$$\text{maximize} \quad z' \tag{MWD}$$

$$\begin{aligned}
\text{subject to} \quad & \sum_{i=1}^{k} z_i^* D_{\uparrow}^2 \Phi(., y_i)(x', z', t - x', r - z')(x - x') \\
& + \sum_{j=1}^{p} w_j^* D_{\uparrow}^2 G_j(x', w_j', t - x', s - w_j')(x - x') \geq 0, \\
& \forall x \in A, \\
& \text{for some } k \in \mathbb{N}, y_i \in B(x'), t \in A, \\
& r \in \Phi(t, y_i) + \mathbb{R}_+, s \in G_j(u) + \mathbb{R}_+, \\
& \sum_{j=1}^{p} w_j^* w_j' \geq 0, \\
& x' \in A, z' = \max \bigcup_{y \in B} \Phi(x', y), \\
& w' = (w_1', w_2', ..., w_p'), w_j' \in G_j(x'), \\
& z^* = (z_1^*, z_2^*, ..., z_k^*), w^* = (w_1^*, w_2^*, ..., w_p^*), \\
& z_i^* \geq 0, w_j^* \geq 0, (1 \leq i \leq k, 1 \leq j \leq p), \\
& \text{and } \sum_{i=1}^{k} z_i^* \neq 0.
\end{aligned}$$

Definition 3.3: A feasible point (x', z', w', z^*, w^*) of the second-order Mond-Weir type dual problem (MWD) is called a maximizer of (MWD) if for all feasible points (x, z, w, z_1^*, w_1^*) of the problem (MWD),

$$z \leq z'.$$

Theorem 3.3: (Second-order weak duality) Let A be a nonempty convex subset of $\mathbb{R}^n$ and $x_0 \in S$. Let (x', z', w', z^*, w^*) be a feasible point of the problem (MWD). Let $t \in A$, $r \in \Phi(t, y_i) + \mathbb{R}_+$, and $s \in G_j(u) + \mathbb{R}_+$. Assume that $\Phi(., y_i)$, $(1 \leq i \leq k)$, is second-order ρ_i-$\mathbb{R}_+$-convex at (x', z') in the direction $(t - x', r - z')$ and G_j, $(1 \leq j \leq p)$ is second-order ρ_j'-$\mathbb{R}_+$-convex (x', w_j') in the direction $(t - x', s - w_j')$, respectively, with respect to 1, on A and Eqn. (III.3) is satisfied. Then,

$$\max \bigcup_{y \in B} \Phi(x_0, y) \not< z'.$$

Proof: We can prove the theorem by the method of contradiction.
Suppose that for $z_0 = \max \bigcup_{y \in B} \Phi(x_0, y)$,

$$z_0 < z'$$

As $z' = \max \bigcup_{y \in B} \Phi(x', y)$, we have

$$z' \in \Phi(x', y_i), \forall i = 1, 2, ..., k.$$

Choose

$$z_i \in \Phi(x_0, y_i), \forall i = 1, 2, ..., k.$$

Again, as $z_0 = \max \bigcup_{y\in B} \Phi(x_0, y)$ and $y_i \in B(x') \subseteq B$, we have

$$z_i \leq z_0.$$

Therefore,

$$z_i \leq z_0 < z'.$$

Hence,

$$\sum_{i=1}^{k} z_i^* z_i < \sum_{i=1}^{k} z_i^* z'.$$

As $x_0 \in S$, there exists $w_j \in G_j(x_0) \cap (-\mathbb{R}_+)$, $(1 \leq j \leq p)$. Since $w_j^* \geq 0$, $(1 \leq j \leq p)$, we have

$$w_j^* w_j \leq 0, \forall j = 1, 2, ..., p.$$

Therefore,

$$\sum_{j=1}^{p} w_j^* w_j \leq \sum_{j=1}^{p} w_j^* w_j'.$$

Hence,

$$\sum_{i=1}^{k} z_i^* z_i + \sum_{j=1}^{p} w_j^* w_j < \sum_{i=1}^{k} z_i^* z' + \sum_{j=1}^{p} w_j^* w_j'. \quad \text{(III.7)}$$

Since $\Phi(., y_i)$, $(1 \leq i \leq k)$, is second-order ρ_i-$\mathbb{R}_+$-convex at (x', z') in the direction $(t - x', r - z')$ and G_j, $(1 \leq j \leq p)$ is second-order ρ_j'-$\mathbb{R}_+$-convex (x', w_j') in the direction $(t - x', s - w_j')$, respectively, with respect to 1, on A, we have

$$\Phi(x_0, y_i) - z' \subseteq D_\uparrow^2 \Phi(., y_i)(x', z', t - x', r - z')(x_0 - x') + \rho_i \|x_0 - x'\|^2 + \mathbb{R}_+$$

and

$$G_j(x_0) - w_j' \subseteq D_\uparrow^2 G_j(x', w_j', t - x', s - w_j')(x_0 - x') + \rho_j' \|x_0 - x'\|^2 + \mathbb{R}_+.$$

Therefore,

$$z_i - z' \subseteq D_\uparrow^2 \Phi(., y_i)(x', z', t - x', r - z')(x_0 - x') + \rho_i \|x_0 - x'\|^2 + \mathbb{R}_+ \quad \text{(III.8)}$$

and

$$w_j - w_j' \subseteq D_\uparrow^2 G_j(x', w_j', t - x', s - w_j')(x_0 - x') + \rho_j' \|x_0 - x'\|^2 + \mathbb{R}_+. \quad \text{(III.9)}$$

From the constraints of (MWD) and Eqs. (III.3), (III.8), and (III.9), we have

$$\sum_{i=1}^{k} z_i^*(z_i - z') + \sum_{j=1}^{p} w_j^*(w_j - w_j') \geq 0,$$

which contradicts (III.7). Hence,

$$z_0 \not< z'.$$

Therefore,

$$\max \bigcup_{y\in B} \Phi(x_0, y) \not< z'.$$

It completes the proof of the theorem. ■

Theorem 3.4: (Second-order strong duality) Let (x', z') be a minimizer of the problem (MP) and $w_j' \in G_j(x') \cap (-\mathbb{R}_+)$, $(1 \leq j \leq p)$. Suppose that for a positive integer k, $z_i^* \geq 0$, $y_i \in B(x')$, $(1 \leq i \leq k)$, with $\sum_{i=1}^{k} z_i^* \neq 0$, and $w_j^* \geq 0$, $(1 \leq j \leq p)$, Eqs. (III.1) and (III.2) are satisfied at (x', z', w', z^*, w^*). Then (x', z', w', z^*, w^*) is a feasible solution of (MWD). If the second-order weak duality Theorem 3.3 holds between the problems (MP) and (MWD), then (x', z', w', z^*, w^*) is a maximizer of (MWD).

Proof: As the Eqs. (III.1) and (III.2) are satisfied at (x', z', w', z^*, w^*), we have

$$\sum_{i=1}^{k} z_i^* D_\uparrow^2 \Phi(., y_i)(x', z', t - x', r - z')(x - x') + \sum_{j=1}^{p} w_j^* D_\uparrow^2 G_j(x', w_j', t - x', s - w_j')(x - x') \geq 0, \forall x \in A$$

and

$$w_j^* w_j' = 0, \forall j = 1, 2, ..., p.$$

Hence, (x', z', w', z^*, w^*) is a feasible solution of (MWD). Assume that the second-order weak duality Theorem 3.3 holds between the problems (MP) and (MWD) and (x', z', w', z^*, w^*) is not a maximizer of (MWD).
Hence there exists a feasible point (x, z, w, z_1^*, w_1^*) of (MWD), such that

$$z' < z.$$

It contradicts the second-order weak duality Theorem 3.3.
Hence, (x', z', w', z^*, w^*) is a maximizer of (MWD). ■

Theorem 3.5: (Second-order converse duality) Let (x', z', w', z^*, w^*) be a feasible point of the problem (MWD) and A be a nonempty convex subset of $\mathbb{R}^n$. Let $t \in A$, $r \in \Phi(t, y_i) + \mathbb{R}_+$, and $s \in G_j(u) + \mathbb{R}_+$. Assume that $\Phi(., y_i)$, $(1 \leq i \leq k)$, is second-order ρ_i-$\mathbb{R}_+$-convex at (x', z') in the direction $(t - x', r - z')$ and G_j, $(1 \leq j \leq p)$ is second-order ρ_j'-$\mathbb{R}_+$-convex (x', w_j') in the direction $(t - x', s - w_j')$, respectively, with respect to 1, on A and Eqn. (III.3) is satisfied. Let $x' \in S$. Then (x', y') is a minimizer of (MP).

Proof: Assume that (x', z') is not a minimizer of (MP). Hence there exist $x \in S$ and $z = \max \bigcup_{y\in B} \Phi(x, y)$, such that

$$z < z'.$$

As $y_i \in B(x')$ $(1 \leq i \leq k)$, we have

$$\max \bigcup_{y\in B} \Phi(x', y) \in \Phi(x', y_i).$$

As $z' = \max \bigcup_{y\in B} \Phi(x', y)$, we have

$$z' \in \Phi(x', y_i), \forall i = 1, 2, ..., k.$$

Choose

$$z_i \in \Phi(x, y_i), \forall i = 1, 2, ..., k.$$

Again, as $z = \max \bigcup_{y\in B} \Phi(x, y)$ and $y_i \in B(x') \subseteq B$, we have

$$z_i \leq z.$$

Therefore,

$$z_i \leq z < z'.$$

Hence,

$$\sum_{i=1}^{k} z_i^* z_i < \sum_{i=1}^{k} z_i^* z'.$$

As $x \in S$, there exists

$$w_j \in G_j(x) \cap (-\mathbb{R}_+), (1 \leq j \leq p).$$

Since $w_j^* \geq 0$, $(1 \leq j \leq p)$, we have

$$w_j^* w_j \leq 0, \forall j = 1, 2, ..., p.$$

So,

$$\sum_{j=1}^{p} w_j^* w_j \leq 0.$$

The constraints of the dual problem (MWD) give

$$\sum_{j=1}^{p} w_j^* w_j' \geq 0.$$

Therefore,

$$\sum_{j=1}^{p} w_j^* w_j \leq \sum_{j=1}^{p} w_j^* w_j'.$$

Hence,

$$\sum_{i=1}^{k} z_i^* z_i + \sum_{j=1}^{p} w_j^* w_j < \sum_{i=1}^{k} z_i^* z' + \sum_{j=1}^{p} w_j^* w_j'. \quad \text{(III.10)}$$

Since $\Phi(., y_i)$, $(1 \leq i \leq k)$, is second-order ρ_i-$\mathbb{R}_+$-convex at (x', z') in the direction $(t - x', r - z')$ and G_j, $(1 \leq j \leq p)$ is second-order ρ_j'-$\mathbb{R}_+$-convex (x', w_j') in the direction $(t - x', s - w_j')$, respectively, with respect to 1, on A, we have

$$\Phi(x, y_i) - z' \subseteq D_\uparrow^2 \Phi(., y_i)(x', z', t - x', r - z')(x - x') + \rho_i \|x - x'\|^2 + \mathbb{R}_+$$

and

$$G_j(x) - w_j' \subseteq D_\uparrow^2 G_j(x', w_j', t - x', s - w_j')(x - x') + \rho_j' \|x - x'\|^2 + \mathbb{R}_+.$$

Therefore,

$$z_i - z' \subseteq D_\uparrow^2 \Phi(., y_i)(x', z', t - x', r - z')(x - x') + \rho_i \|x - x'\|^2 + \mathbb{R}_+ \quad \text{(III.11)}$$

and

$$w_j - w_j' \subseteq D_\uparrow^2 G_j(x', w_j', t - x', s - w_j')(x - x') + \rho_j' \|x - x'\|^2 + \mathbb{R}_+. \quad \text{(III.12)}$$

From the constraints of (MWD) and Eqs. (III.3), (III.11), and (III.12), we have

$$\sum_{i=1}^{k} z_i^*(z_i - z') + \sum_{j=1}^{p} w_j^*(w_j - w_j') \geq 0.$$

It contradicts (III.10).
Consequently, (x', z') is a minimizer of the problem (MP). ■

C. Second-order Wolfe type dual

We consider a second-order Wolfe type dual (WD) of the problem (MP), where $\Phi(., y_i)$ and G_j are second-order contingent epiderivable set-valued maps, where $y_i \in B(x')$ and $x' \in A$.

$$\text{maximize} \quad z' + \sum_{j=1}^{p} w_j^* w_j' \quad (WD)$$

$$\begin{aligned} \text{subject to} \quad & \sum_{i=1}^{k} z_i^* D_\uparrow^2 \Phi(., y_i)(x', z', t - x', r - z')(x - x') \\ & + \sum_{j=1}^{p} w_j^* D_\uparrow^2 G_j(x', w_j', t - x', s - w_j')(x - x') \geq 0, \\ & \forall x \in A, \\ & \text{for some } k \in \mathbb{N}, y_i \in B(x'), t \in A, \\ & r \in \Phi(t, y_i) + \mathbb{R}_+, s \in G_j(u) + \mathbb{R}_+, \\ & x' \in A, z' = \max \bigcup_{y \in B} \Phi(x', y), \\ & w' = (w_1', w_2', ..., w_p'), w_j' \in G_j(x'), \\ & z^* = (z_1^*, z_2^*, ..., z_k^*), w^* = (w_1^*, w_2^*, ..., w_p^*), \\ & z_i^* \geq 0, w_j^* \geq 0, (1 \leq i \leq k, 1 \leq j \leq p), \\ & \text{and } \sum_{i=1}^{k} z_i^* \neq 0. \end{aligned}$$

Definition 3.4: A feasible point (x', z', w', z^*, w^*) of the second-order Wolfe type dual problem (WD) is called a maximizer of (WD) if for all feasible points $(x, z, w, \overline{z}^*, \overline{w}^*)$ of the problem (WD),

$$z + \sum_{j=1}^{p} \overline{w}_j^* w_j \leq z' + \sum_{j=1}^{p} w_j^* w_j',$$

where $w' = (w_1', w_2', ..., w_p')$, $w^* = (w_1^*, w_2^*, ..., w_p^*)$, $w = (w_1, w_2, ..., w_p)$, and $\overline{w}^* = (\overline{w}_1^*, \overline{w}_2^*, ..., \overline{w}_p^*) \in \mathbb{R}^p$.
We can prove the Wolfe type duality results for the problem (MP). The proofs are very similar to those of Theorems 3.3 - 3.5, and hence we omit.

Theorem 3.6: (Second-order weak duality) Let A be a nonempty convex subset of $\mathbb{R}^n$ and $x_0 \in S$. Let (x', z', w', z^*, w^*) be a feasible point of the problem (WD). Let $t \in A$, $r \in \Phi(t, y_i) + \mathbb{R}_+$, and $s \in G_j(u) + \mathbb{R}_+$. Assume that $\Phi(., y_i)$, $(1 \leq i \leq k)$, is second-order ρ_i-$\mathbb{R}_+$-convex at (x', z') in the direction $(t - x', r - z')$ and G_j, $(1 \leq j \leq p)$ is second-order ρ_j'-$\mathbb{R}_+$-convex (x', w_j') in the direction $(t - x', s - w_j')$, respectively, with respect to 1, on A and Eqn. (III.3) is satisfied. Then,

$$\max \bigcup_{y \in B} \Phi(x_0, y) \not< z' + \sum_{j=1}^{p} w_j^* w_j'.$$

Theorem 3.7: (Second-order strong duality) Let (x', z') be a minimizer of the problem (MP) and $w_j' \in G_j(x') \cap (-\mathbb{R}_+)$, $(1 \leq j \leq p)$. Suppose that for a positive integer k, $z_i^* \geq 0$, $y_i \in B(x')$, $(1 \leq i \leq k)$, with $\sum_{i=1}^{k} z_i^* \neq 0$, and $w_j^* \geq 0$, $(1 \leq j \leq p)$, Eqs. (III.1) and (III.2) are satisfied at (x', z', w', z^*, w^*). Then (x', z', w', z^*, w^*) is a feasible solution of (WD). If the second-order weak duality Theorem 3.6 holds between the problems (MP) and (WD), then (x', z', w', z^*, w^*) is a maximizer of (WD).

Theorem 3.8: (Second-order converse duality) Let (x', z', w', z^*, w^*) be a feasible point of the problem (WD), with $\sum_{j=1}^{p} w_j^* w_j' \geq 0$, and A be a nonempty convex subset of $\mathbb{R}^n$. Let $t \in A$, $r \in \Phi(t, y_i) + \mathbb{R}_+$, and $s \in G_j(u) + \mathbb{R}_+$. Assume that $\Phi(., y_i)$, $(1 \leq i \leq k)$, is second-order ρ_i-$\mathbb{R}_+$-convex at (x', z') in the direction $(t - x', r - z')$ and G_j, $(1 \leq j \leq p)$ is second-order ρ_j'-$\mathbb{R}_+$-convex (x', w_j') in the direction $(t - x', s - w_j')$, respectively, with respect to 1, on A and Eqn. (III.3) is satisfied. Let $x' \in S$. Then (x', y') is a minimizer of (MP).

D. Second-order mixed type dual

We consider a second-order mixed type dual (MD) of the problem (MP), where $\Phi(., y_i)$ and G_j are second-order contingent epiderivable set-valued maps, where $y_i \in B(x')$ and $x' \in A$.

$$\text{maximize} \quad z' + \sum_{j=1}^{p} w_j^* w_j' \qquad (MD)$$

$$\text{subject to} \quad \sum_{i=1}^{k} z_i^* D_\uparrow^2 \Phi(., y_i)(x', z', t - x', r - z')(x - x') + \sum_{j=1}^{p} w_j^* D_\uparrow^2 G_j(x', w_j', t - x', s - w_j')(x - x') \geq 0,$$

$\forall x \in A$,

for some $k \in \mathbb{N}, y_i \in B(x'), t \in A$,

$r \in \Phi(t, y_i) + \mathbb{R}_+, s \in G_j(u) + \mathbb{R}_+$,

$\sum_{j=1}^{p} w_j^* w_j' \geq 0, x' \in A, z' = \max \bigcup_{y \in B} \Phi(x', y)$,

$w' = (w_1', w_2', ..., w_p'), w_j' \in G_j(x')$,

$z^* = (z_1^*, z_2^*, ..., z_k^*), w^* = (w_1^*, w_2^*, ..., w_p^*)$,

$z_i^* \geq 0, w_j^* \geq 0, (1 \leq i \leq k, 1 \leq j \leq p)$,

and $\sum_{i=1}^{k} z_i^* \neq 0$.

Definition 3.5: A feasible point (x', z', w', z^*, w^*) of the second-order mixed type dual problem (MD) is said to be a maximizer of (MD) if for all feasible points $(x, z, w, \overline{z}^*, \overline{w}^*)$ of the problem (MD),

$$z + \sum_{j=1}^{p} \overline{w}_j^* w_j \leq z' + \sum_{j=1}^{p} w_j^* w_j',$$

where $w' = (w_1', w_2', ..., w_p')$, $w^* = (w_1^*, w_2^*, ..., w_p^*)$, $w = (w_1, w_2, ..., w_p)$, and $\overline{w}^* = (\overline{w}_1^*, \overline{w}_2^*, ..., \overline{w}_p^*) \in \mathbb{R}^p$.
We can prove the mixed type duality results of the problem (MP). The proofs are very similar to those of Theorems 3.3 - 3.5, and hence we omit.

Theorem 3.9: (Second-order weak duality) Let A be a nonempty convex subset of $\mathbb{R}^n$ and $x_0 \in S$. Let (x', z', w', z^*, w^*) be a feasible point of the problem (MD). Let $t \in A$, $r \in \Phi(t, y_i) + \mathbb{R}_+$, and $s \in G_j(u) + \mathbb{R}_+$. Assume that $\Phi(., y_i)$, $(1 \leq i \leq k)$, is second-order ρ_i-$\mathbb{R}_+$-convex at (x', z') in the direction $(t - x', r - z')$ and G_j, $(1 \leq j \leq p)$ is second-order ρ_j'-$\mathbb{R}_+$-convex (x', w_j') in the direction $(t - x', s - w_j')$, respectively, with respect to 1, on A and Eqn. (III.3) is satisfied. Then,

$$\max \bigcup_{y \in B} \Phi(x_0, y) \not< z' + \sum_{j=1}^{p} w_j^* w_j'.$$

Theorem 3.10: (Second-order strong duality) Let (x', z') be a minimizer of the problem (MP) and $w_j' \in G_j(x') \cap (-\mathbb{R}_+)$, $(1 \leq j \leq p)$. Assume that for a positive integer k, $z_i^* \geq 0$, $y_i \in B(x')$, $(1 \leq i \leq k)$, with $\sum_{i=1}^{k} z_i^* \neq 0$, and $w_j^* \geq 0$, $(1 \leq j \leq p)$, Eqs. (III.1) and (III.2) are satisfied at (x', z', w', z^*, w^*). Then (x', z', w', z^*, w^*) is a feasible solution of (MD). If the second-order weak duality Theorem 3.9 holds between the problems (MP) and (MD), then (x', z', w', z^*, w^*) is a maximizer of (MD).

Theorem 3.11: (Second-order converse duality) Let (x', z', w', z^*, w^*) be a feasible point of (MD) and A be a nonempty convex subset of $\mathbb{R}^n$. Let $t \in A$, $r \in \Phi(t, y_i) + \mathbb{R}_+$, and $s \in G_j(u) + \mathbb{R}_+$. Assume that $\Phi(., y_i)$, $(1 \leq i \leq k)$, is second-order ρ_i-$\mathbb{R}_+$-convex at (x', z') in the direction $(t - x', r - z')$ and G_j, $(1 \leq j \leq p)$ is second-order ρ_j'-$\mathbb{R}_+$-convex (x', w_j') in the direction $(t - x', s - w_j')$, respectively, with respect to 1, on A and Eqn. (III.3) is satisfied. Let $x' \in S$. Then (x', y') is a minimizer of (MP).

IV. Conclusions

In this paper, we develop the second-order KKT sufficient conditions of a set-valued minimax programming problem (MP) under second-order contingent epiderivative for set-valued maps. The duals of second-order Mond-Weir type, Wolfe type, and mixed type are formulated for the problem (MP), and the corresponding duality results are developed via second-order ρ-cone convexity assumption.

Acknowledgement: The authors are very thankful to referees for their valuable comments which improved the presentation of the paper.

References

Aubin JP (1981) Contingent derivatives of set-valued maps and existence of solutions to nonlinear inclusions and differential inclusions. in: Mathematical Analysis and Applications, Part A, L Nachbin (ed), Academic Press, New York pp 160–229

Aubin JP, Frankowska H (1990) Set-Valued Analysis. Birhäuser, Boston

Bector CR, Bhatia BL (1985) Sufficient optimality conditions and duality for a minmax problem. Util Math 27:229–247

Bector CR, Chandra S, Husain I (1992) Sufficient optimality conditions and duality for a continuous-time minmax programming problem. Asia-Pacific J Oper Res 9:55–76

Borwein J (1977) Multivalued convexity and optimization: a unified approach to inequality and equality constraints. Math Program 13(1):183–199

Bram J (1966) The Lagrange multiplier theorem for max-min with several constraints. SIAM J Appl Math 14:665–667

Cambini A, Martein L, Vlach M (1999) Second order tangent sets and optimality conditions. Math Japonica 49(3):451–461

Chandra S, Kumar V (1995) Duality in fractional minimax programming. J Austral Math Soc (Ser A) 58:376–386

Corley HW (1987) Existence and Lagrangian duality for maximizations of set-valued functions. J Optim Theory Appl 54(3):489–501

Danskin JM (1966) The theory of max-min with applications. SIAM J Appl Math 14:641–644

Danskin JM (1967) The Theory of Max-Min and Its Applications to Weapons Allocation Problems. Springer-Verlag, Berlin

Das K, Nahak C (2014) Sufficient optimality conditions and duality theorems for set-valued optimization problem under generalized cone convexity. Rend Circ Mat Palermo 63(3):329–345

Das K, Nahak C (2015) Sufficiency and duality of set-valued optimization problems via higher-order contingent derivative. J Adv Math Stud 8(1):137–151

Das K, Nahak C (2016a) Optimality conditions for approximate quasi efficiency in set-valued equilibrium problems. SeMA J 73(2):183–199

Das K, Nahak C (2016b) Set-valued fractional programming problems under generalized cone convexity. Opsearch 53(1):157–177

Das K, Nahak C (2017a) Approximate quasi efficiency of set-valued optimization problems via weak subdifferential. SeMA J 74(4):523–542

Das K, Nahak C (2017b) Optimization problems with difference of set-valued maps under generalized cone convexity. J Appl Math Inform 35(1–2):147–163

Das K, Nahak C (2017c) Set-valued minimax programming problems under generalized cone convexity. Rend Circ Mat Palermo 66(3):361–374

Das K, Nahak C (2020) Optimality conditions for set-valued minimax fractional programming problems. SeMA J 77(2):161–179

Datta N, Bhatia D (1984) Duality for a class of nondifferentiable mathematical programming problems in complex space. J Math Anal Appl 101:1–11

Demyanov VF, Malozehon VN (1974) Introduction to Minmax. Wiley, New York

Jahn J, Rauh R (1997) Contingent epiderivatives and set-valued optimization. Math Method Oper Res 46(2):193–211

Jahn J, Khan A, Zeilinger P (2005) Second-order optimality conditions in set optimization. J Optim Theory Appl 125(2):331–347

Li SJ, Teo KL, Yang XQ (2008a) Higher-order Mond–Weir duality for set-valued optimization. J Comput Appl Math 217(2):339–349

Li SJ, Teo KL, Yang XQ (2008b) Higher-order optimality conditions for set-valued optimization. J Optim Theory Appl 137(3):533–553

Mehra A, Bhatia D (1999) Optimality and duality for minmax problems involving arcwise connected and generalized arcwise connected functions. J Math Anal Appl 231(2):425–445

Schmitendorf WE (1977) Necessary conditions and sufficient conditions for static minmax problems. J Math Anal Appl 57:683–693

Tanimoto S (1981) Duality for a class of nondifferentiable mathematical programming problems. J Math Anal Appl 79:286–294

Zalmai GJ (1985) Optimality conditions for a class of continuous-time minmax programming problems. Research Report, Department of Mathematics, Washington State Univ., Pullman, WA

Zhu SK, Li SJ, Teo KL (2014) Second-order Karush-Kuhn-Tucker optimality conditions for set-valued optimization. J Glob Optim 58(4):673–692