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Development of a Discrete Probability Distribution and its application to the Pattern of Child Deaths

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Abstract: **The Infant and child mortality is known as a good and sensitive indicator of development of a nation and impact of government intervention programs and policies. High rate of death before first birthday particularly in the first month of the birth represents the gross reproductive loss of physical, economical and psychological resources of the females. Child death has been major concern of the researchers and demographers because of its apparent relationship with the level of fertility and indirect relationship with the acceptance of modern contraceptive means. The study of mortality especially child death within the age limit 0-4 years is one of the important area of population sciences. Its higher value indicates the lower level of success of health and intervention programs. In the present study an attempt has been to discuss and propose a single parameter discrete probability model to study the pattern of child deaths. The parameters involved in the model under consideration have been estimated with maximum likelihood. Real data sets on child mortality have been used to talk about the applicability and validity of the models. The suitability of models has been also checked by -2loglikelihood, AIC and Chi-square test.**

Index Terms: **Child death, Probability distribution, Maximum likelihood estimates**

I. INTRODUCTION

High rate of death before first birth day particularly in the first month of the birth represents the gross reproductive loss of physical, economical and psychological resources of the females. If one can identify the responsible biological and socio-demographic factor for child death, the health and intervention program can be reformulated and implemented to reduce the intensity of child death. The Infant and child mortality is known as a good and sensitive indicator of development of a nation and impact of government intervention programs and policies. Effective control in reduction of mortality has been one of the remarkable achievements across the world. Child death has been major concern of the researchers and demographers because of its apparent relationship with the level of fertility and indirect relationship with the acceptance of modern contraceptive means. Child death has positive impact on fertility (Bhuyan et al. 1996) and reduction in child death or enhance in the probability of survival of children is one of the known reasons of reduction in the fertility.

Studies on early age mortality in the last five decades are mostly confined to infant death, but, it has been realized that child death also need to be examined in addition to infant mortality. A number of parametric models for the study of the age patterns of mortality have been developed over the year. First parametric model for duration of mortality was developed by Gompertz (1825). The very first attempt to study infant mortality through parametric modeling was proposed by Keyfitz (1977) who used a hyperbolic function for the study infant mortality. An attempt to represent mortality across the entire age range was the eight parameter non-linear model of Heligman and Pollard (1980). Later, Hartman (1982) proposed a logarithmic approximation and Weibull function was recommended by Cheo (1981). After that many attempt have been made to study age at infant mortality through mathematical models. In these circumstances, a number of attempts have been made to study the age pattern of mortality mathematical model (Krishnan and Jin, 1993 and Chauhan, 1997). Bhuyan & Deogratias (1999) used modified Polya-Aeppli and Poisson-Gamma distribution, Singh et al. (2011) and Singh et al. (2012) proposed Beta-Binomial model and inflated Binomial distribution for the number of child death for fixed parity i.e. fixed number of children ever born. Further Singh et al. (2012) developed a model by mixing Poisson distribution with the Beta-Binomial distribution to study the variation of child death in the society. Since child death to a female is a discrete type data therefore we need to develop discrete distribution.

II. A BRIEF DESCRIPTION OF DISCRETE PROBABILITY DISTRIBUTION

Modeling of discrete data is used in several fields such as health sciences, epidemiology, applied science, sociology, psychology

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and agricultural sciences. Several discrete distributions are proposed especially for the over-dispersed data. However, it was create that the traditional discrete distributions e.g. Binomial, Poisson, Geometric and Negative Binomial distribution have limited use and fail some times to capture variability of the data. Since numerous real count data show either over-dispersion, where the variance is more than mean or under-dispersion, where the variance is smaller than mean. Therefore it is important to develop some discrete distributions for the situation discussed. It is observed that in real life many phenomenon provide discrete data. Therefore, it is essential to develop a reasonable and suitable discrete model for these situations. A lot of new continuous distributions are available but researchers paid a little attention in the development of discrete distributions.

Therefore, there is a need to focus on more realistic discrete distributions (Rezaei Roknabadi et al. 2009). One of the methods to develop discrete distribution is discretization of a continuous distribution (Lai, 2013). The discrete Weibull distribution was proposed by Nakagawa and Osaki (1975) and the discrete Rayleigh distribution, has been studied by Roy (2002). Other newly developed distributions analogues to continuous distribution, discrete gamma emerge to received important applications. It is used first by Yang (1994) in the area of molecular biology. A discrete analogue of the Burr and Pareto distributions was developed by Krishna and Pundir (2009) and Aghababaei Jaziet et al. (2010) proposed a discrete inverse Weibull distribution. Also Gómez-Déniz (2010) constructs a discrete generalized exponential distribution using Marshall and Olkin method (1997). Again Gómez-Déniz and Calderín-Ojeda (2011) proposed discrete Lindley distributions. The aim of this paper is to introduce a new one-parameter discrete distribution. This model is shown to perform better than the traditional model.

III. Probability Distribution for the Number of Child Deaths

Negative Binomial Distribution

This is a mixture of Poisson and Gamma distribution, where the unknown parameter λ of Poisson distribution follows gamma distribution. Let *x* denote the number of child death to a female follows Poisson distribution with parameter λ, which represents the expected number of child death to the females but this is not constant for each and every females and we assume that it follows Gamma distribution. Bhuyan & Deogratias (1999) in his paper used a form of Poisson-Gamma distribution given by Evans (1953) as

$$p(x;m,a) = (1+a)^{-\left(\frac{m}{a}\right)}\left(\frac{a}{1+a}\right)^{x}\overline{\left|\left(\frac{m}{a}+x\right)\right.}\Big/\underline{|x}\overline{\left|\left(\frac{m}{a}\right)\right.} \quad ; \quad m>0, a>0 \tag{1}$$

where $x = 0, 1, 2 \ldots\ldots\ldots\ldots n$

The parameters *m* and *a* are estimated by $\hat{m} = k_1$ and $\hat{a} = \frac{k_2}{k_1} - 1$, where k_1 and k_2 are first two cumulants which is used by Bhuyan & Deogratias (1999). In the above equation (1) if $\frac{m}{a} = r$ and $\frac{1}{1+a} = p$ then equation (1) can be written as

$$p(x;r,p) = {}^{x+r-1}c_x q^x p^r \quad ; \; 0 < p < 1 \; and \; q = 1-p$$

Proposed Distributions

We know that the pdf of Negative Binomial Distribution is as follows

$p(x;r,p) = {}^{x+r-1}c_x q^x p^r$. The expression of mean and variance is given below

$$E(x) = \frac{rq}{p} \qquad Var(x) = \frac{rq}{p^2}$$

Now, if $r = 1$ the distribution becomes Geometric distribution as follows

$$p_1(x) = q^x p$$

The mean and variance is as follows

$$E(x) = \frac{q}{p} \qquad Var(x) = \frac{q}{p^2}$$

Again, if $r = 2$ then the distribution becomes

$$p_2(x) = {}^{x+1}c_x p^2 q^x = (x+1) q^x p^2$$

and the mean and variance of this distribution is

$$E(x) = \frac{2q}{p} \qquad Var(x) = \frac{2q}{p^2}$$

Now a mixture of the above two distributions as is proposed as

Let $p(x) = \alpha p_1(x) + (1-\alpha) p_2(x)$

Where the mixing proportion is $\alpha = \frac{p}{1+p}$ and $1-\alpha = \frac{1}{1+p}$

therefore, $p(x) = \left(\frac{p}{1+p}\right) q^x p + \left(\frac{1}{1+p}\right)(x+1) q^x p^2$

Thus the proposed distribution is a single parameter (*p*) distribution and is given as

$$p(x) = \frac{p^2 q^x}{1+p}(x+2) \tag{2}$$

Moment Generating and Characteristic Function

$$M_x(t) = \sum_{x=0}^{\infty} e^{tx} p(x) = \sum_{x=0}^{\infty} e^{tx} \frac{p^2 q^x}{(1+p)}(x+2)$$

$$= \frac{p^2}{(1+p)} \sum_{x=0}^{\infty} (qe^t)^x (x+2) = \frac{p^2}{(1+p)}\left[\sum_{x=0}^{\infty} x(qe^t)^x + 2\sum_{x=0}^{\infty} (qe^t)^x\right]$$

$$= \frac{p^2}{(1+p)}\left[qe^t \sum_{x=0}^{\infty} x(qe^t)^{x-1} + 2\sum_{x=0}^{\infty} (qe^t)^x\right]$$

$$= \frac{p^2}{(1+p)}\left[qe^t\left(1-qe^t\right)^{-2} + 2\left(1-qe^t\right)^{-1}\right]$$

Using the formula $\sum_{x=0}^{\infty} x a^{x-1} = (1-a)^{-2}$ and $\sum_{x=0}^{\infty} a^x = (1-a)^{-1}$

Therefore,

$$M_x(t) = \frac{p^2}{(1+p)}\left[\frac{qe^t}{\left(1-qe^t\right)^2} + \frac{2}{\left(1-qe^t\right)}\right] = \frac{p^2\left(2-qe^t\right)}{(1+p)\left(1-qe^t\right)^2} \tag{3}$$

For getting different moments we differentiate equation (3) with respect to t in and equating zero we get the $E(x)$ of the distribution.

$$M'_x(t) = \frac{p^2}{(1+p)}\left[\frac{2qe^t}{\left(1-qe^t\right)^3} + \frac{qe^t}{\left(1-qe^t\right)^2}\right]$$

$$M'_x(t)\big|_{t=0} = \frac{p^2}{(1+p)}\left[\frac{2q}{p^3}+\frac{q}{p^2}\right] = \frac{p^2}{(1+p)}\frac{q}{p^3}(2+p)$$

$$= \frac{q(2+p)}{(p+p^2)} = \frac{2-p-p^2}{p+p^2} = E(x)$$

This provides the average number of failure before first success. It is clear that if p the probability of success is increasing then $E(x)$ is decreasing. Now replacing t with it in equation (3) we get the characteristic function as

$$\phi_x(t) = \frac{p^2\left(2-qe^{it}\right)}{(1+p)\left(1-qe^{it}\right)^2} \quad (4)$$

IV. Estimation of Parameter

Zero Cell Frequency Method

We know the pmf of the proposed distribution is as $p(x) = \frac{p^2q^x}{1+p}(x+2)$

Let x is zero then the $p(0) = \frac{2p^2}{1+p}$ (5)

we may get an estimate p from this equation easily.

Recurrence Relation

We know the pmf of the proposed distribution is as $p(x) = \frac{p^2q^x}{1+p}(x+2)$ and $p(x+1) = \frac{p^2q^{x+1}}{1+p}(x+3)$

Therefore $\frac{p(x+1)}{p(x)} = \frac{q(x+3)}{(x+2)} \Rightarrow p(x+1) = \frac{(1-p)(x+3)p(x)}{(x+2)}$, using estimate of p and $p(0)$, we can get $p(1)$, $p(2)$ and so on.

Let x=0 then $1-p = \frac{2p(1)}{3p(0)} = \frac{2f_1}{3f_0}$, where f_0 and f_1 are the frequencies of zero[th] and first cell. This may be used as estimation of p.

Moments through Recurrence Relation

Now we have from the recurrence relation $(x+2)p(x+1) = q(x+3)p(x)$

$$\Rightarrow (x+1)p(x+1) + p(x+1) = qxp(x) + 3qp(x) \quad (6)$$

In order to get the first moment M_1, we are taking summation both side then we have

$$\Rightarrow (x+1)\sum_{x=0}^{\infty} p(x+1) + \sum_{x=0}^{\infty} p(x+1) = q\sum_{x=0}^{\infty} xp(x) + 3q\sum_{x=0}^{\infty} p(x)$$

$$\Rightarrow M_1 + 1 - p_0 = qM_1 + 3q; \quad \text{where } M_1 = \sum_{x=0}^{\infty} xp(x) \text{ or } \sum_{x=0}^{\infty}(x+1)p(x+1)$$

$$\Rightarrow M_1 = \frac{3q - 1 + \left(\frac{2p^2}{1+p}\right)}{1-q} = \frac{(2-3p)(1+p)+2p^2}{p(1+p)}$$

$$\Rightarrow M_1 = \frac{2-p-p^2}{p(1+p)} = \frac{(2+p)(1-p)}{p(1+p)} \quad (7)$$

Now multiplying (x+1) in the above equation (4) the we have

$$(x+1)^2 p(x+1) + (x+1)p(x+1) = qx(x+1)p(x) + 3q(x+1)p(x)$$

$$\Rightarrow (x+1)^2 p(x+1) + (x+1)p(x+1) = qx^2 p(x) + 4qxp(x) + 3qp(x)$$

In order to get the second moment M_2, we are taking summation both side then we have

$$M_2 + M_1 = qM_2 + 4qM_1 + 3q$$

$\Rightarrow M_2 = \frac{4qM_1 - M_1 + 3q}{1-q}$, putting M_1 from the equation (7) we get

$$\Rightarrow M_2 = \frac{(3-4p)(2+p)(1-p)+p(1+p)(3-3p)}{p^2(1+p)}$$

$$\Rightarrow M_2 = \frac{(1-p)[(3-4p)(2+p)+3p(1+p)]}{p^2(1+p)}$$

$$\Rightarrow M_2 = \frac{(1-p)(6-2p-p^2)}{p^2(1+p)} \quad (8)$$

similarly we can get the higher order moments.

Method of Moments

Now the first moment of the proposed distribution is

$$E(x) = \sum_{x=0}^{\infty} x.\frac{p^2q^x}{1+p}[x+2] = \sum_{x=0}^{\infty} x^2.\frac{p^2q^x}{1+p} + \sum_{x=0}^{\infty} 2x.\frac{p^2q^x}{1+p}$$

$$= \frac{p^2}{1+p}\sum_{x=0}^{\infty} x^2.q^x + \frac{2p^2}{1+p}\sum_{x=0}^{\infty} x.q^x = \frac{p^2}{1+p}\left[\sum_{x=0}^{\infty} x^2.q^x + 2\sum_{x=0}^{\infty} x.q^x\right]$$

and the second moment is

$$E(x^2) = \sum_{x=0}^{\infty} x^2.\frac{p^2q^x}{1+p}[x+2] = \sum_{x=0}^{\infty} x^3.\frac{p^2q^x}{1+p} + \sum_{x=0}^{\infty} 2x^2.\frac{p^2q^x}{1+p}$$

$$= \frac{p^2}{1+p}\sum_{x=0}^{\infty} x^3.q^x + \frac{2p^2}{1+p}\sum_{x=0}^{\infty} x^2.q^x = \frac{p^2}{1+p}\left[\sum_{x=0}^{\infty} x^3.q^x + 2\sum_{x=0}^{\infty} x^2.q^x\right]$$

after solving both series, we get,

$$E(x) = \frac{(2+p)(1-p)}{p(1+p)} \quad (9)$$

Similarly we can obtain $E(x^2)$ as

$$E(x^2) = \frac{(6-2p-p^2)(1-p)}{p^2(1+p)} \quad (10)$$

Thus the variance is $V(x) = \frac{(2+4p)(1-p)}{p^2(1+p)^2}$ (11)

Now we have Fisher's Index of dispersion is as follows

$$\gamma = \frac{V(x)}{E(x)} = \frac{2+4p}{p(1+p)(2+p)}$$

If $V(x) > E(x)$ then

$$(2+4p) > p(1+p)(2+p) \Rightarrow (2+4p) > p(2+p)$$

$$\Rightarrow 2+4p > 2p+p^2 \Rightarrow p^2 - 2p - 2 < 0$$

$$\Rightarrow p^2 - 2p + 1 - 3 < 0 \Rightarrow (1-p)^2 < 3 \quad \Rightarrow q^2 < 3$$

which is true, thus $V(x) > E(x)$. Therefore, it is clear that Fisher's index of dispersion is more than 1 means the proposed distribution is a over dispersed distribution.

Maximum Likelihood Estimate

Suppose $X_1, X_2, X_3, \ldots\ldots X_n$ be an independent and identical distributed (iid) random variables of size n with pmf (2). Then, the likelihood function based on observed sample $X = \{x_1, x_2, x_3, \ldots\ldots, x_n\}$ is defined as

$$p(x) = \frac{p^2 q^x}{1+p}(x+2)$$

$$L = \prod_{i=1}^{n} p(x_i) \Rightarrow \prod_{i=1}^{n} \frac{p^2 q^{x_i}}{1+p}(x_i+2) = \frac{p^{2n} q^{\sum_{i=1}^{n} x_i}}{(1+p)^n} \prod_{i=1}^{n}(x_i+2) \quad (12)$$

Taking log both sides we get

$$\log L = 2n \log p - n \log(1+p) + \log q \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \log(x_i+2)$$

Now differentiate with respect to p both sides we get

$$\frac{\partial \log L}{\partial p} = \frac{2n}{p} - \frac{n}{1+p} - \frac{\sum_{i=1}^{n} x_i}{1-p} = 0 \quad \text{since } q = 1-p$$

$$\Rightarrow \frac{\sum_{i=1}^{n} x_i}{1-p} = \frac{2n}{p} - \frac{n}{1+p} \quad \Rightarrow \frac{\sum_{i=1}^{n} x_i}{n} = (1-p)\left[\frac{2+2p-p}{p(1+p)}\right]$$

$$\Rightarrow \bar{x} = \frac{(2+p)(1-p)}{p(1+p)} = \frac{2-p-p^2}{p+p^2} \quad (13)$$

This is similar to the first moment i.e. $E(X)$.

$$\Rightarrow \frac{2-p-p^2}{p+p^2} + 1 = (\bar{x}+1) \Rightarrow \frac{p+p^2}{2} = \frac{1}{(\bar{x}+1)} = k(say)$$

$p + p^2 - 2k = 0 \Rightarrow p = \frac{-1 \pm \sqrt{1+8k}}{2}$. Since parameter p can't be negative, hence $p = \frac{-1+\sqrt{1+8k}}{2}$

$$\Rightarrow \hat{p} = \frac{1}{2}\left(\sqrt{1+\frac{8}{(\bar{x}+1)}} - 1\right) \quad (14)$$

V. Application of the Model

To illustrate the application and comparison of the models discussed above, the real data used by Bhuyan & Deogratias (1999) has been considered. We know that the Poisson distribution is a suitable model for the situations where events seem to occur at random and rare in the nature such as the number of customers arriving at a service point, the number of telephone calls arriving at an exchange in a certain time period. In the real life problems, the occurrences of successive events are heterogeneous thus the Negative Binomial distribution is a possible alternative to the Poisson distribution (Johnson et al. 1992). Further, for fitting Poisson distribution the mean and variance should be equal and in case of Poisson-Gamma distribution mean should be less than the variance. In real life problems, most of all count data have more variability, thus they can only described by an over dispersed count data model. The proposed distribution is an over dispersed distribution. Since Poisson-Gamma distribution having two parameters however the proposed distributions having only one parameter and is simple in mathematical complexity. From the Table 1, the ML estimate of proposed distribution is 0.749 however, it is for negative binomial distribution is m=0.525 and a=0.418 and the value of χ^2 with p-value is 1.55 (0.46) and 1.43 (0.70) for proposed and negative binomial distribution respectively. On the basis of p-value, -2loglikelihood and AIC, it is reveal that the proposed distribution gives better fit than negative binomial distribution and thus it can be considered as an alternative way for modeling child death data.

Table 1: Observed and Expected distribution of females according to the number of child deaths

No. of Child Deaths	Observed No. of females	Expected No. of females	
		Negative Binomial Distribution	Proposed Distribution
0	805	807.08	803.23
1	306	299.08	302.42
2	93	99.52	101.21
3	36	31.86	31.75
4	7	9.99	9.56
5	2	3.10	2.80
6	1	0.95	0.80
7	2	0.42	0.23
Total	1252	1252.00	1252.00
Parameters		m =0.525, a =0.418	p =0.749
χ^2 after pooling		1.55	1.43
degree of freedom		2	3
p-value		0.46	0.70
-2loglikelihood		2457.93	2458.49
AIC		2461.93	2460.49

I would like to show the suitability of proposed distribution using another data set taken from Simon, L. J. (1961) and found the proposed distribution is an excellent alternative of the negative binomial distribution (Table 2).

Table 2: Observed and Expected distribution of data on number of accident (taken from Simon, L. J.; 1961)

No. of Child Deaths	Observed No. of females	Expected No. of females	
		Negative Binomial Distribution	Proposed Distribution
0	99	95.86	97.57
1	65	75.83	74.02
2	57	50.35	49.91
3	35	31.30	31.55
4	20	18.79	19.15
5	10	11.04	11.30
6	4	6.40	6.53
7	0	3.67	3.72
8	3	2.08	2.09
9	4	1.18	1.16
10	0	0.66	0.64
11	1	0.84	0.35
Total	298	298.00	298.00
Parameters		m =1.708, a =1.159	p =0.494
χ^2 after pooling		4.06	3.67
degree of freedom		5	6
p-value		0.54	0.72
-2loglikelihood		1057.54	1057.26
AIC		1061.54	1059.26

Table 3 provides the summary of finding based on the parameter of the proposed distribution for the child death data. Average child death in the first group is 0.335 however in second group it is

0.670. The weighted average for first group is 0.144 and for second group is 0.383. Second group contribute 73 percent of child deaths and only 27 percent child death contributed by first group. This proposed model split the population in two groups and identified the higher risk group. Program makers may use this model to plan their strategies and starts some intervention program to reduce child mortality in the high risk group.

Table 3: Summary of findings on the basis of parameter *p* of the proposed distribution

	First Group	Second Group	
Proportion	0.43	0.57	*Theoretical Average child death is 0.527*
Average child death	0.335	0.670	
Weighted Average child death	0.144	0.383	*Empirical Average child death is 0.526*
Percent contribution in child death	0.27	0.73	

VI. Conclusion

Since the proposed distribution is a mixture of two distributions which is generated from negative binomial distribution. Therefore the proposed distribution is more flexible than the traditional negative binomial distribution. Also the proposed distribution is a single parameter distribution thus this is simpler in nature in terms of mathematical complexity than negative binomial distribution. Therefore the proposed distribution may be used as an alternative of Poisson-gamma (negative binomial), Poisson-exponential and Poisson-Lindley distribution.

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