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A New Approach to Define Algebraic Structure and Some Homomorphism Functions on Set of Pythagorean Triples and Set of Reciprocal Pythagorean Triples

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Abstract: In this paper, focused to study Algebraic Structure and some Homomorphism functions On Set of Pythagorean Triples $\boldsymbol{P_T} = \{(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})\epsilon \boldsymbol{Z}^3 : \boldsymbol{x}^2 + \boldsymbol{y}^2 = \boldsymbol{z}^2\}$ and Set of Reciprocal Pythagorean triples $\boldsymbol{RP_T} = \left\{(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})\epsilon \boldsymbol{Z}^3 : \frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{z^2}\right\}$ under the binary operation of usual multiplications. For some

$$\boldsymbol{P_1} = (x_1, y_1, z_1)\epsilon \boldsymbol{P_T}, \boldsymbol{P_2} = (x_2, y_2, z_2)\epsilon \boldsymbol{P_T} \text{ with } \quad p_1 . p_2 = \begin{Bmatrix} (|y_1y_2 - x_1x_2|, x_1y_2 + x_2y_1, z_1z_2) & [\boldsymbol{Lemma\ A}] \\ (x_1x_2, y_1z_2 + y_2z_1, y_1y_2 + z_1z_2) & [\boldsymbol{Lemma\ B}] \end{Bmatrix}.$$

Also we are proven Every Pythagorean Triple (x, y, z) is having corresponding Reciprocal Pythagorean Triple in the form of (xz, yz, xy) *and vice versa*. Apply this corollary to define Algebraic Structure on Set of Reciprocal Pythagorean Triples. Also applied above binary operations of usual multiplication on Set of Sequence of Fibonacci type numbers to generate some subsets of Set of Pythagorean triples. Also focused to study some Homomorphism functions on Set of Pythagorean triples and Set of Reciprocal Pythagorean Triples. Also, we are proven some Properties of Trigonometric Ratio's and Compound Angles for Reciprocal Pythagorean Triples.

Index Terms: **Algebraic Structure, Binary Operation, Homomorphism functions, Pythagorean theorem, Reciprocal Pythagorean theorem.**

I. INTRODUCTION

The solutions to the quadratic Diophantine equation $a^2 + b^2 = c^2$ are given by Pythagorean theorem and corresponding Reciprocal Pythagorean Triples (a , b , h) is $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{h^2}$ with c = $\frac{ab}{h}$,here 'h' is a altitude, which are can be used in the computation of the area of a triangle.

P_1	P_2	$(\lvert y_1y_2 - x_1x_2\rvert, x_1y_2 + x_2y_1, z_1z_2)$
(5,12,13)	(4,3,5)	(16,63,65)
(7,24,25)	(3,4,5)	(75,100,125)
(4,3,5)	(8,15,17)	(13,84,85)
(4,3,5)	(4,3,5)	(8,24,25)
(3,4,5)	(8,15,17)	(36,77,85)
(1,0,1)	(8,15,17)	(8,15,17)
(1,0,1)	(1,0,1)	(1,0,1)

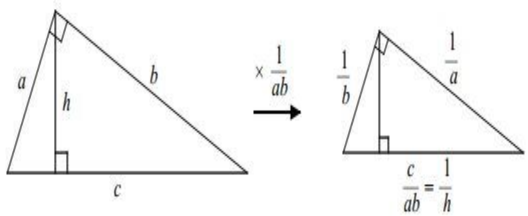


Figure 1. Pythagorean & Reciprocal Pythagorean Theorem

In this paper we are focused to study Algebraic Structure and some Homomorphism Functions defined on Set of Pythagorean triples and Set of Reciprocal Pythagorean triples.

A. *Algebraic Structure of Set of Pythagorean Triples*

1). Lemma A: Introduce to define the binary operation of Usual multiplication '.'

On Set of Pythagorean triples is $P_1.P_2 = (|y_1y_2 - x_1x_2|, x_1y_2 + x_2y_1, z_1z_2)$ For some $P_1 = (x_1, y_1, z_1)\epsilon P_T, P_2 = (x_2, y_2, z_2)\epsilon P_T$.

Proof: Consider $(y_1y_2 - x_1x_2)^2 + (x_1y_2 + x_2y_1)^2 = (x_1x_2)^2 + (y_1y_2)^2 + (x_1y_2)^2 + (x_2y_1)^2 = [x_1^2 + y_1^2][x_2^2 + y_2^2] = z_1^2z_2^2$.

It follows that $(|y_1y_2 - x_1x_2|, x_1y_2 + x_2y_1, z_1z_2)$ is becomes to a Pythagorean triple. Hence the binary operation of usual multiplication is well defined on Set of Pythagorean triples. Some examples are represented in below table

Table 1: verification of Lemma A, By choosing some $P_1\epsilon P_T, P_2\epsilon P_T$

2). Lemma B: Introduce to define another binary operation of usual multiplication '.' On Set of Pythagorean triples is $P_1.P_2 = (x_1x_2, y_1z_2 + y_2z_1, y_1y_2 + z_1z_2)$

Proof: Consider $(y_1y_2 + z_1z_2)^2 - (y_1z_2 + y_2z_1)^2 = (y_1y_2)^2 + (z_1z_2)^2 - (y_1z_2)^2 - (y_2z_1)^2 = z_2^2[z_1^2 - y_1^2] - y_2^2[z_1^2 - y_1^2] = [z_1^2 - y_1^2][z_2^2 - y_2^2] = x_1^2x_2^2$.

It follows that $(x_1x_2)^2 + (y_1z_2 + y_2z_1)^2 = (y_1y_2 + z_1z_2)^2$ implies that $(x_1x_2, y_1z_2 + y_2z_1, y_1y_2 + z_1z_2)$is becomes to Pythagorean triple. Some examples are represented in below table.

Table 2: verification of Lemma B , By choosing some $P_1\epsilon P_T, P_2\epsilon P_T$

$P_1 = (x_1, y_1, z_1)$	$P_2 = (x_2, y_2, z_2)$	$(x_1x_2, y_1z_2 + y_2z_1, y_1y_2 + z_1z_2)$
(5,12,13)	(4,3,5)	(20,99,101)
(7,24,25)	(3,4,5)	(21,220,221)
(4,3,5)	(8,15,17)	(32,126,130)
(4,3,5)	(4,3,5)	(16,30,34)
(3,4,5)	(8,15,17)	(24,143,145)
(1,0,1)	(8,15,17)	(8,15,17)
(1,0,1)	(1,0,1)	(1,0,1)

Hence the binary operation of usual multiplication'.' Is well defined on Set of Pythagorean triples. i.e. for some $P_1 = (x_1, y_1, z_1)\epsilon P_T, P_2 = (x_2, y_2, z_2)\epsilon P_T$ with $p_1.p_2 = \begin{cases} (|y_1y_2 - x_1x_2|, x_1y_2 + x_2y_1, z_1z_2) & [Lemma\ A] \\ (x_1x_2, y_1z_2 + y_2z_1, y_1y_2 + z_1z_2) & [Lemma\ B] \end{cases}$. Apply little bit of calculations of Lemma A and Lemma B, we can verify easily Commutative axiom $(p_1.p_2 = p_2.p_1)$ and Associative Axioms $(p_1.p_2).p_3 = p_1.(p_2.p_3)$are satisfies the elements of Set of Pythagorean TriplesP_T with existence of Identity element (1,0,1). It proves that Set of Pythagorean triples can form as commutative Monoid. Now we can go to introduce to study some Injective homomorphism functions on Set of Pythagorean triples

II. Some Homomorphism Functions on Set of Pythagorean Triples

Case 1: $\emptyset_o$ *is an* injective homomorphism mapping with respect to Lemma B, defined as $\emptyset_o : \{(2x-1)/x\epsilon N\} \rightarrow Z^3(P_T)$.Foreach odd integer x,$\emptyset_o(x) = \left(x, \frac{x^2-1}{2}, \frac{x^2+1}{2}\right)$.

Proof: From Reference [1], for each odd integer x ,$\left(x, \frac{x^2-1}{2}, \frac{x^2+1}{2}\right)$is becomes to Pythagorean triple. Consider $\emptyset_o(x_1).\emptyset_o(x_2)$= $\left(x_1, \frac{{x_1}^2-1}{2}, \frac{{x_1}^2+1}{2}\right).\left(x_2, \frac{{x_2}^2-1}{2}, \frac{{x_2}^2+1}{2}\right)$.

Now we can apply Lemma B, obtain that $\left(x_1x_2, \frac{(x_1x_2)^2-1}{2}, \frac{(x_1x_2)^2+1}{2}\right) = \emptyset_o(x_1.x_2)$. It follows that $\emptyset_o(x)$ is becomes to Homomorphism function under the binary operation of Lemma B. Some examples are represented in below table.

Table 3: $\emptyset_o(x)$ *is an* injective homomorphism for each odd integerx_1,x_2

x	x	$\emptyset_o(x_1)$	$\emptyset_o(x_2)$	$\emptyset_o(x_1).\emptyset_o(x$	$\emptyset_o(x_1.x_2)$
		(1,0,1)	(3,4,5)	(3,4,5)	(3,4,5)
		(3,4,5)	(3,4,5)	(9,40,41)	(9,40,41)
		(3,4,5)	(5,12,13)	(15,112,113)	(15,112,113)
		(3,4,5)	(7,24,25)	(21,220,221)	(21,220,221)
		(5,12,13)	(7,24,25)	(35,612,613)	(35,612,613)

Case 2: $\emptyset_e(x)$ is injective Homomorphism mapping with respect to Lemma B, defined as $\emptyset_e : \{(2x)/x\epsilon N\} \rightarrow Z^3(P_T)$. For even integer x,$\emptyset_e(x) = \left(x, \left(\frac{x}{2}\right)^2 - 1, \left(\frac{x}{2}\right)^2 + 1\right)$.

Proof: Clearly (2,0,2)∈ $\emptyset_e$ implies that $\emptyset_e(x)$is non empty set, also from Reference [1], each even integer x, $\left(x, \left(\frac{x}{2}\right)^2 - 1, \left(\frac{x}{2}\right)^2 + 1\right)$ is become to Pythagorean triple. Consider $\emptyset_e(x_1).\emptyset_e(x_2)$=$\left(x_1, \left(\frac{x_1}{2}\right)^2 - 1, \left(\frac{x_1}{2}\right)^2 + 1\right).\left(x_2, \left(\frac{x_2}{2}\right)^2 - 1, \left(\frac{x_2}{2}\right)^2 + 1\right)$.

Now we can apply Lemma B, obtain $\left(x_1x_2, 2\left(\frac{(x_1x_2)^2}{16} - 1\right), 2\left(\frac{(x_1x_2)^2}{16} + 1\right)\right)$ =2$\emptyset_e\left(\frac{x_1.x_2}{2}\right)$. It follows that $\emptyset_e(x)$ is become to

Homomorphism function for each even integer x, under above binary operation of Lemma B.

x_1	x_2	$\emptyset_e(x_1)$	$\emptyset_e(x_2)$	$2\emptyset_e(x_1.x_2/2)$	$\emptyset_e(x_1).\emptyset_e(x_2)$
		(4,3,5)	(4,3,5)	(16,30,34)	(16,30,34)
		(6,8,10)	(4,3,5)	(24,70,74)	(24,70,74)
		(6,8,10)	(6,8,10)	(36,160,164)	(36,160,164)
		(8,15,17)	(6,8,10)	(48,286,290)	(48,286,290)
		(4,3,5)	(8,15,17)	(32,126,130)	(32,126,130)

Table 4: $\emptyset_e : \{(2x)/x \epsilon N\} \to Z^3(P_T)$choose even positive integers x_1, x_2

Case3: $\emptyset_e(x_1 . x_2) = \emptyset_e(x_1).\emptyset_o(x_2) = \emptyset_o(x_1).\emptyset_e(x_2)$, is injective homomorphism function, under the binary operation [B].

Proof: Consider $\emptyset_e(x_1).\emptyset_o(x_2)=\left(x_1,\left(\frac{x_1}{2}\right)^2-1,\left(\frac{x_1}{2}\right)^2+1\right).\left(x_2,\frac{{x_2}^2-1}{2},\frac{{x_2}^2+1}{2}\right)$ Apply Lemma B, obtains $\left(x_1x_2,\left(\frac{x_1x_2}{2}\right)^2-1,\left(\frac{x_1x_2}{2}\right)^2+1\right)=\emptyset_e(x_1 . x_2)$

Table 5: $\emptyset_e : \emptyset_e(x_1 . x_2) = \emptyset_e(x_1).\emptyset_o(x_2)$ for even integer x_1 *and odd integer* x_2

x_1	x_2	$\emptyset_e(x_1)$	$\emptyset_o(x_2)$	$\emptyset_e(x_1).\emptyset_o(x_2)$	$\emptyset_e(x_1.x_2)$
		(4,3,5)	(3,4,5)	(12,35,37)	(12,35,37)
		(6,8,10)	(3,4,5)	(18,80,82)	(18,80,82)
		(6,8,10)	(5,12,13)	(30,224,226)	(30,224,226)
		(8,15,17)	(7,24,25)	(56,783,785)	(56,783,785)
		(4,3,5)	(7,24,25)	(28,195,197)	(28,195,197)

B)*Corollary 1: Introduce to define some subset of Pythagorean Triples with using sequence of Fibonacci numbers as follows.*

Let $\emptyset : Z^2 \to Z^3(P_T)$ with $\emptyset(F_n, F_{n+1}) = ((2F_{n+1}(F_n + F_{n+1}), F_n(2F_{n+1} + F_n), F_{n+1}^2 + (F_n + F_{n+1})^2)$ is becomes to subset of Set of Pythagorean Triple also this subset becomes to semi group under the binary operation$(x_1, y_1, z_1).(x_2, y_2, z_2) = (x_1x_2, y_1z_2 + y_2z_1, y_1y_2 + z_1z_2)$for all $(x_1, y_1, z_1) \in \emptyset, (x_2, y_2, z_2) \in \emptyset$

Proof: Fibonacci Numbers$\{1,1,2,3,5,8,13,21 \ldots\ldots\}$ Satisfies following Recurrence Relation $F_n = F_{n-1} + F_{n-2}$ $for\ n \geq 2, with\ F_0 = 1, F_1 = 1$. Also Introduce to define subset of Set Of Pythagorean Triple is $(2(F_{n+1}(F_n + F_{n+1}), F_n(2F_{n+1} + F_n), F_{n+1}^2 + (F_n + F_{n+1})^2)$.

p	q	X $=(p^2+q^2)(p^2-q^2)$	Y= $2pq(p^2+q^2)$	z= $2pq(p^2-q^2)$	Reciprocal Pythagorean Triples
2	1	15	20	12	(15,20,12)
3	1	80	60	48	(80,60,48)
3	2	65	156	60	(65.156,60)
4	1	255	136	120	(255,136,120)
4	2	240	320	192	(240,320,192)
4	3	175	600	168	(175,600,168)
5	1	624	260	240	(624,260,240)

Table 6: Subset of Set of Pythagorean Triples for sequence of Fibonacci Numbers

F_n	F_{n+1}	$(2(F_{n+1}(F_n + F_{n+1}), F_n(2F_{n+1} + F_n), F_{n+1}^2 + (F_n + F_{n+1})^2)$
1	1	(4,3,5)
1	2	(12,5,13)
2	3	(30,16,34)
3	5	(80,39,89)
5	8	(208,105,233)

Let $\emptyset : Z^2 \to Z^3(P_T)$ with $\emptyset(F_n, F_{n+1}) = (2(F_{n+1}(F_n + F_{n+1}), F_n(2F_{n+1} + F_n), F_{n+1}^2 + (F_n + F_{n+1})^2)$. Apply Little Bit of Calculation by replacing n values , $\emptyset(F_n, F_{n+1})$is becomes to Algebraic Structure of Semi Group (Satisfies Closure and Associative axiom)under the binary operation $(x_1, y_1, z_1).(x_2, y_2, z_2) = (x_1x_2, y_1z_2 + y_2z_1, y_1y_2 + z_1z_2)$for all $(x_1, y_1, z_1) \in \emptyset, (x_2, y_2, z_2) \in \emptyset$.

Corollary 2: Introduce to define some subset of Pythagorean Triples with using sequence of Pell numbers as follows

Let $\varphi: Z^2 \to Z^3(P_T)$ with $\varphi(P_n, P_{n+1}) = (2P_nP_{n+1}, P_{n+1}^2 - P_n^2, P_{n+1}^2 + P_n^2)$.is becomes to subset of Set of Pythagorean Triple also this subset becomes to semi group under the binary operation$(x_1, y_1, z_1).(x_2, y_2, z_2) = (x_1x_2, y_1z_2 + y_2z_1, y_1y_2 + z_1z_2)$for all $(x_1, y_1, z_1) \in \emptyset, (x_2, y_2, z_2) \in \emptyset$

Proof: Pell Numbers$\{0,1,2,5,12,29 \ldots\ldots,\}$ Satisfies following Recurrence Relation $P_n = 2P_{n-1} + P_{n-2}\ for\ n \geq 2,\ with\ P_0 = 0, P_1 = 1$. Also introduce to define subset of Set of Pythagorean Triple using pill numbers is($(2P_nP_{n+1}, P_{n+1}^2 - P_n^2, P_{n+1}^2 + P_n^2)$.

Table 7: Subset of Set of Pythagorean Triples for sequence of Pell Numbers

P_n	P_{n+1}	$(2P_nP_{n+1}, P_{n+1}^2 - P_n^2, P_{n+1}^2 + P_n^2)$
0	1	(0,1,1)
1	2	(4,3,5)
2	5	(20,21,29)
5	12	(120,119,169)
12	29	(696,697,985)

Let $\varphi : \boldsymbol{Z^2} \to Z^3(\boldsymbol{P_T})$ with$\varphi(P_n, P_{n+1}) = (2P_nP_{n+1}, P_{n+1}^2 - P_n^2, P_{n+1}^2 + P_n^2)$.Apply Little Bit of Calculation by replacing n values , $\emptyset(P_n, P_{n+1})$is become to Algebraic Structure of Semi

Group (Satisfies Closure and Associative axiom)under the binary operation $(x_1, y_1, z_1).(x_2, y_2, z_2) = (x_1x_2, y_1z_2 + y_2z_1, y_1y_2 + z_1z_2)$for all $(x_1, y_1, z_1) \in \emptyset, (x_2, y_2, z_2) \in \emptyset$.

m	x	X_1	Y_1	Z_1	Reciprocal Pythagorean Triple	$\left(\frac{x(x^2+1)}{2}, \frac{(x^2-1)(x^2+1)}{4}, \frac{x(x^2-1)}{2}\right)$
1	3	15	20	12	(15,20,12)	(15,20,12)
2	5	65	156	60	(65, 156, 60)	(65, 156, 60)
3	7	175	600	168	(175, 600, 168)	(175, 600, 168)
4	9	369	1640	360	(369,1640, 360)	(369, 1640, 360)
5	11	671	3660	660	(671,3660, 660)	(671, 3660, 660)

III. Algebraic Structure and some Homomorphic functions on Set of Reciprocal Pythagorean Triples

First, we can introduce some subsets of Set of Reciprocal Pythagorean Triples $RP_T = \left\{(x, y, z)\epsilon Z^3 : \frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{z^2}\right\}$with using following Theorems.

Theorem 1: For any two integers p, q (with p > q), (x, y, z) = $((p^2+q^2)(p^2-q^2),\ 2pq(p^2+q^2), 2pq(p^2-q^2))$ is becomes to the Reciprocal Pythagorean Triple

Proof: Consider Reciprocal Pythagorean Theorem $\frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{z^2}$. It will follow $\left(\frac{x}{z} + \frac{x}{y}\right)\left(\frac{x}{z} - \frac{x}{y}\right) = 1$. For some positive integers p,

$(x_1, y_1, z_1)\ \epsilon RP_T$	$(x_1z_1, y_1z_1, x_1y_1)\ \epsilon P_T$
(20, 15, 12)	(240,180,300) = $[(240)^2 + (180)^2 = (300)^2]$
(60, 80, 48)	(2880,3840,4800) $=[(2880)^2 + (3880)^2 = (4800)^2]$
(136, 255, 120)	(16320,30600,34680) $= [(16320)^2 + (30600)^2 = (34680)^2]$
(260, 624, 240)	(62400,149760,162240) $= [62400^2 + (149760)^2 = (162240)^2]$
(444, 1295, 420)	(186480,543900,574980) $= [186480^2 + (543900)^2 = (574980)^2]$

q (with p > q), assume $\frac{x}{z} + \frac{x}{y} = \frac{p}{q}$, $\frac{x}{z} - \frac{x}{y} = \frac{q}{p}$. Which is implies $\frac{2x}{z} = \frac{p}{q} + \frac{q}{p}$, $\frac{2x}{y} = \frac{p}{q} - \frac{q}{p}$, follows that $2x = z\left(\frac{p}{q} + \frac{q}{p}\right)$, $2x = y\left(\frac{p}{q} - \frac{q}{p}\right)$. If we can choose the value of x= $(p^2+q^2)(p^2-q^2)$ then z = $2pq(p^2-q^2)$, y = $2pq(p^2+q^2)$. It is Simple and effective Method to find Integer Solution for Reciprocal Pythagorean Theorem. Clearly $\frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{((p^2+q^2)(p^2-q^2))^2} + \frac{1}{(2pq(p^2+q^2))^2} = \frac{1}{(2pq(p^2-q^2))^2} = \frac{1}{z^2}$. Some examples are represented in below table.

Table 8: Reciprocal Pythagorean Triples For Some Positive Integers P, Q With (P>Q)

Theorem 2:

$Rpt_o(x) = \left\{\left(\frac{x(x^2+1)}{2}, \frac{(x^2-1)(x^2+1)}{4}, \frac{x(x^2-1)}{2} : x \text{ is an odd integer}\right)\right\}$

is as subset of Set of Reciprocal Pythagorean Triple.

Proof: Clearly $\frac{1}{\left(\frac{x(x^2+1)}{2}\right)^2} + \frac{1}{\left(\frac{(x^2-1)(x^2+1)}{4}\right)^2} = \frac{1}{\left(\frac{x(x^2-1)}{2}\right)^2}$

For generalization, odd integer numbers are denoted by 2 m + 1 for some m = 1, 2 , 3,….. It follows that Reciprocal Pythagorean Triples are represented by $Rpt_o(m) = \left\{\left(\frac{(2m+1)((2m+1)^2+1)}{2}, \frac{((2m+1)^4-1)}{4}, \frac{(2m+1)((2m+1)^2-1)}{2}\right) : m = 1,2,3,\dots\dots\right\}$ $Rpt_o(m) = \left\{\left((2m+1)(2m^2+2m+1), 4m^4 + 8m^3 + 6m^2 + 2m, (2m+1)(2m^2+2m)\right) : m = 1,2,3,\dots\dots\right\}$

Table 8: Some examples are represented in below table

m	x	X_1	Y_1	Z_1	Reciprocal Pythagorean Triple	$\left(x\left(\left(\frac{x}{2}\right)^2+1\right)\right), \left(\left(\frac{x}{2}\right)^2-1\right)\left(\left(\frac{x}{2}\right)^2+1\right), x\left(\left(\frac{x}{2}\right)^2-1\right)$
2	4	20	15	12	(20, 15, 12)	(20, 15, 12)
3	6	60	80	48	(60, 80, 48)	(60, 80, 48)
4	8	136	255	120	(136,255, 120)	(136, 255, 120)
5	10	260	624	240	(260,624, 240)	(260, 624, 240)
6	12	444	1295	420	(444,295, 420)	(444, 1295, 420)

Table 9: Some examples are represented in below table

Theorem 3: $Rpt_e(x) = \left\{\left(x\left(\left(\frac{x}{2}\right)^2+1\right)\right), \left(\left(\frac{x}{2}\right)^2-1\right)\left(\left(\frac{x}{2}\right)^2+1\right), x\left(\left(\frac{x}{2}\right)^2-1\right) : x \text{ is an even number greater than } 2\right\}$is asubset of Set of Reciprocal Pythagorean Triple.

Proof: Clearly$\frac{1}{\left(x\left(\left(\frac{x}{2}\right)^2+1\right)\right)^2} + \frac{1}{\left(\left(\left(\frac{x}{2}\right)^2-1\right)\left(\left(\frac{x}{2}\right)^2+1\right)\right)^2} = \frac{1}{\left(x\left(\left(\frac{x}{2}\right)^2-1\right)\right)^2}$ For generalization even integer numbers are denoted by 2 m for some m= 2, 3, … It follows that Reciprocal Pythagorean Triples are represented by $Rpt_e(x) = \left\{\left(2m(m^2+1)\right), ((m)^4 - 1), 2m((m)^2-1) : m = 2,3,\dots\dots\dots\right\}$.

Corollary 3: If $(x_1, y_1, z_1)\ \epsilon RP_T$ then $(x_1z_1, y_1z_1, x_1y_1)\ \epsilon P_T$ and vice versa.

Proof: Let $(x_1, y_1, z_1)\ \epsilon RP_T$ implies that $\frac{1}{(x_1)^2} + \frac{1}{(y_1)^2} = \frac{1}{(z_1)^2}$.

Consider $(x_1z_1)^2 + (y_1z_1)^2 = (z_1)^2[{x_1}^2 + {y_1}^2] = (x_1y_1)^2$.

It follows that f $(x_1, y_1, z_1)\ \epsilon RP_T$ then $(x_1z_1, y_1z_1, x_1y_1)\ \epsilon P_T$ and vice versa. Table 10: Some examples are represented in below table.

Lemma C: For some $P_1^1 = (x_1, y_1, z_1)\ \epsilon RP_T, P_2^1 = (x_2, y_2, z_2)\epsilon RP_T$, define corresponding Binary Operation on RP_T with using of Corollary 3 and Lemma A is

$P_1^1.P_2^1 = (x_1, y_1, z_1).(x_2, y_2, z_2)\ (x_1z_1, y_1z_1, x_1y_1).(x_2z_2, y_2z_2, x_2y_2)$
$=\left(|y_1y_2z_1z_2 - x_1x_2z_1z_2|(x_1x_2y_1y_2), (x_1y_2z_1z_2 + y_1x_2z_1z_2)(x_1x_2y_1y_2), (y_1y_2z_1z_2 - x_1x_2z_1z_2)(x_1y_2z_1z_2 + y_1x_2z_1z_2)\right)$

Corollary 4: $(x_1z_1, y_1z_1, x_1y_1)\ \epsilon P_T, (x_2z_2, y_2z_2, x_2y_2)\ \epsilon P_T$ Then $(x_1z_1, y_1z_1, x_1y_1).(x_2z_2, y_2z_2, x_2y_2) = (x_1z_1x_2z_2, y_1z_1x_2y_2 + y_2z_2x_1y_1, y_1z_1y_2z_2 + x_1y_1x_2y_2)$

Proof: Similar Proof of Lemma B and corollary 3, above Binary Operation is well defined on Set of Pythagorean triples.

Lemma D: For some $P_1^1 = (x_1, y_1, z_1)\ \epsilon RP_T, P_2^1 = (x_2, y_2, z_2) \epsilon RP_T$, From Corollary 3, Corollary 4, we can go to define corresponding Binary Operation on RP_T is

$P_1^1.P_2^1 = (x_1, y_1, z_1).(x_2, y_2, z_2) = (x_1z_1, y_1z_1, x_1y_1).(x_2z_2, y_2z_2, x_2y_2)$ $= ((x_1z_1x_2z_2)(y_1z_1y_2z_2 + x_1y_1x_2y_2), (y_1z_1x_2y_2 + y_2z_2x_1y_1)(y_1z_1y_2z_2 + x_1y_1x_2y_2), (x_1z_1x_2z_2)(y_1z_1x_2y_2 + y_2z_2x_1y_1))$

Now we can apply little bit of calculations of above Binary Operations of Lemma C and Lemma Don Set of Reciprocal Pythagorean Triples to form as Algebraic Structure of Semi Group (Satisfies closure axiom and Associative Axiom).

Homomorphism Functions of Set Of Reciprocal Pythagorean Triples

Case 4:Rpt_o is an injective homomorphism mapping defined as $Rpt_o: \{2x + 1\} \to Z^3(RP_T)$with$Rpt_o(x)$= $\left\{\left(\frac{x(x^2+1)}{2}, \frac{(x^2-1)(x^2+1)}{4}, \frac{x(x^2-1)}{2}\right) : x \text{ is odd}\right\}$ under the binary Operation of Lemma D.

Proof: Clearly (1 ,0 ,1)∈ $Rpt_o(x)$. So it is non empty and Also FromTheorem1,$\left(\frac{x(x^2+1)}{2}, \frac{(x^2-1)(x^2+1)}{4}, \frac{x(x^2-1)}{2} : \right)$ X is an odd integer is an element of Set of Reciprocal Pythagorean Triple. we can apply Little Bit of Calculation $Rpt_o(x)$with corollary 3, corollary 4 and Lemma D, it becomes to Homomorphism Injective mapping. i. e $Rpt_o(x_1).Rpt_o(x_2) = Rpt_o(x_1.x_2)$.

Case 5:$Rpt_e(x)$ is an injective mapping defined as $Rpt_e: \{(2x)/x\epsilon N\} \to Z^3(RP_T)$ with

$Rpt_o(x)$= $\left\{\left\{\left(x\left(\left(\frac{x}{2}\right)^2+1\right)\right), \left(\left(\frac{x}{2}\right)^2-1\right)\left(\left(\frac{x}{2}\right)^2+1\right), x\left(\left(\frac{x}{2}\right)^2 - 1\right) : x \text{ is an even}\right\}\right\}$under the binary operationof Lemma D.

Proof: Clearly (2 ,0 ,2)∈ $Rpt_e(x)$ and, $\left(\left(x\left(\left(\frac{x}{2}\right)^2+1\right)\right), \left(\left(\frac{x}{2}\right)^2 - 1\right)\left(\left(\frac{x}{2}\right)^2+1\right), x\left(\left(\frac{x}{2}\right)^2-1\right) : x \text{ is even integer}\right)$ is a element of Set of Reciprocal Pythagorean we can apply Little Bit of Calculation $Rpt_e(x)$ is becomes to Homomorphism mappin:g $Rpt_e(x_1).Rpt_e(x_2) = Rpt_e(x_1.x_2)$ with using of .From Corollary 3, Corollary 4 and Lemma D.

Case 6: Rpt is an injective mapping defined as $Rpt: z^2 \to Z^3(RP_T)$ with

$Rpt(p, q)=\{(p^2 + q^2)(p^2 - q^2),\ 2pq(p^2 + q^2), 2pq(p^2 - q^2)\}$

Proof: Clearly (2 ,0 ,2)∈ $Rpt(p, q)$. So it is non empty and Also from Theorem 1, for some integers p, q (p>q) , $((p^2 + q^2)(p^2 - q^2),\ 2pq(p^2 + q^2), 2pq(p^2 - q^2))$ is a element of Set of Reciprocal Pythagorean Triple and from Corollary 2, Corollary 3 , we can apply Little Bit of Calculation $Rpt(p, q)$ is becomes to Homomorphism. That is $Rpt(p_1, q_1).Rpt(p_2, q_2)$= $Rpt(p_1p_2, q_1q_2)$.

IV. SOME INHERENT PROPERTIES OF RECIPROCAL PYTHAGOREAN TRIPLES

Theorem 4: If (x, y, z) are Reciprocal Pythagorean Triples Then $\frac{yz}{x}$ is Perfect Square.

Proof: we know that x= $(p^2 + q^2)(p^2 - q^2)$,y = $2pq(p^2 + q^2)$, z = $2pq(p^2 - q^2)$.

Consider $\frac{yz}{x} = \frac{(2pq(p^2+q^2))(2pq(p^2-q^2))}{(p^2+q^2)(p^2-q^2)}$=$(2pq)^2$. Hence for some values of p, q, we are proven $\frac{yz}{x}$is Perfect Square.

Table 11: Now we can verify above result by taking some examples.

Reciprocal Pythagorean Triples	$\frac{yz}{x}$
(15,20,12)	$\frac{20*12}{15} = 16 = 4^2$
(80,60,48)	$\frac{60*48}{80} = 36 = 6^2$
(65,156,60)	$\frac{156*60}{65} = 144 = 12^2$
(255,136,120)	$\frac{136*120}{255} = 64 = 8^2$
(240,320,192)	$\frac{320*192}{240} = 256 = 16^2$
(175,600,168)	$\frac{600*168}{175} = 576 = 24^2$
(624,260,240)	$\frac{260*240}{624} = 100 = 10^2$

Theorem 5: If (x, y, z) are Reciprocal Pythagorean Triples Then $\frac{y+z}{y-z}$ is Perfect Square if and only if q divides p.

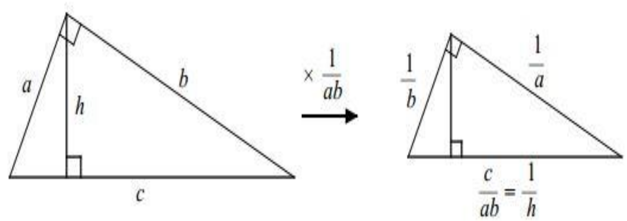
Proof: we know that x= $(p^2 + q^2)(p^2 - q^2)$,y = $2pq(p^2 + q^2)$, z = $2pq(p^2 - q^2)$.

Table12: Some examples are represented in below table.

Reciprocal Pythagorean Triples	$\frac{y+z}{y-z}$
(15,20,12)	$\frac{20+12}{20-12}$ = 4 = 2^2
(80,60,48)	$\frac{60+48}{60-48}$ = 9 =3^2
(65,156,60)	$\frac{156+60}{156-60}$ = 2.25 = 1.5^2
(255,136,120)	$\frac{136+120}{136-120}$ = 16 = 4^2
(240,320,192)	$\frac{320+192}{320-192}$ = 4 = 2^2
(175,600,168)	$\frac{600+168}{600-168}$ = 1.7777778
(624,260,240)	$\frac{260+240}{260-240}$ =25 = 5^2

Consider $\frac{y+z}{y-z} = \frac{2pq(p^2+q^2)+2pq(p^2-q^2)}{2pq(p^2+q^2)-2pq(p^2-q^2)} = \frac{4p^3q}{4q^3p} = \left(\frac{p}{q}\right)^2$. Hence for some values of p, q, we are proven $\frac{y+z}{y-z}$ is Perfect Square if and only if q divides p. Now we can verify above result by taking some examples. Some Trigonometric Functions Relations for

Reciprocal Pythagorean Triples. Now we can go to discuss about Trigonometric functions for Reciprocal Pythagorean Triples (a , b , h) are defined as follows.



Without loss of generality of Pythagorean and Reciprocal Pythagorean theorems ($a^2 + b^2 = c^2$ and $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{h^2}$with c = $\frac{ab}{h}$). Define Trigonometric functions for Reciprocal Pythagorean triples as follows $\sin_{RPT}(\theta_1)= \frac{h}{a}$, $\sin_{RPT}(\theta_2)= \frac{h}{b}$, $\cos_{RPT}(\theta_1)= \frac{h}{b}$, $\cos_{RPT}(\theta_2)= \frac{h}{a}$, $\mathrm{Tan}_{RPT}(\theta_1)= \frac{b}{a}$, $\mathrm{Tan}_{RPT}(\theta_2)= \frac{a}{b}$ with $\theta_1 + \theta_2 = \frac{\pi}{2}$.

Above definitions are Satisfies following axioms of Trigonometric ratios and compound Angles for Reciprocal Pythagorean Triples.

Lemma E: $\mathrm{Sin}^2{}_{RPT}(\theta_1) + \mathrm{Sin}^2{}_{RPT}(\theta_2) = 1$

Proof: Consider $\mathrm{Sin}^2{}_{RPT}(\theta_1) + \mathrm{Sin}^2{}_{RPT}(\theta_2) = \left(\frac{h}{a}\right)^2 + \left(\frac{h}{b}\right)^2 = h^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right) = h^2 * \frac{1}{h^2} = 1$

We can easily verify the result by taking some Reciprocal Pythagorean Triple.

Lemma F: $\mathrm{Sin}^2{}_{RPT}(\theta_1) + \cos^2{}_{RPT}(\theta_1) = 1$

Proof: Consider $\mathrm{Sin}^2{}_{RPT}(\theta_1) + \cos^2{}_{RPT}(\theta_1) = \left(\frac{h}{a}\right)^2 + \left(\frac{h}{b}\right)^2 = h^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right) = h^2 * \frac{1}{h^2} = 1$

We can easily verify the result by taking some reciprocal Pythagorean Triple.

Lemma G: $\mathrm{Sin}^2{}_{RPT}(\theta_2) + \cos^2{}_{RPT}(\theta_2) = 1$

Proof: Consider $\mathrm{Sin}^2{}_{RPT}(\theta_2) + \cos^2{}_{RPT}(\theta_2) = \left(\frac{h}{b}\right)^2 + \left(\frac{h}{a}\right)^2 = h^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right) = h^2 * \frac{1}{h^2} = 1$. We can easily verify the result by taking some reciprocal Pythagorean Triple.

Lemma H: $\mathrm{Cos}^2{}_{RPT}(\theta_1) + \cos^2{}_{RPT}(\theta_2) = 1$

Proof: Consider $\mathrm{Cos}^2{}_{RPT}(\theta_1) + \cos^2{}_{RPT}(\theta_2) = \left(\frac{h}{b}\right)^2 + \left(\frac{h}{a}\right)^2 = h^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right) = h^2 * \frac{1}{h^2} = 1$. We can easily verify the result by taking some reciprocal Triple.

Lemma I: $\mathrm{Cos}^2{}_{RPT}(\theta_1) + \sin^2{}_{RPT}(\theta_2) = 1$

Proof: Consider $\mathrm{Cos}^2{}_{RPT}(\theta_1) + \sin^2{}_{RPT}(\theta_2) = \left(\frac{h}{b}\right)^2 + \left(\frac{h}{a}\right)^2 = h^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right) = h^2 * \frac{1}{h^2} = 1$. We can easily verify the result by taking some reciprocal Triple.

Similarly, we can define Reciprocal Trigonometric functions for Reciprocal Triples defined as follows.

$\mathrm{Cosec}_{RPT}(\theta_1)= \frac{a}{h}$, $\mathrm{Sec}_{RPT}(\theta_1)= \frac{b}{h}$, $\mathrm{Cot}_{RPT}(\theta_1)=\frac{a}{b}$, $\mathrm{Cosec}_{RPT}(\theta_2)= \frac{b}{h}$, $\mathrm{Sec}_{RPT}(\theta_2)= \frac{a}{h}$, $\mathrm{Cot}_{RPT}(\theta_2)= \frac{b}{a}$

Lemma J: $\mathrm{Cosec}^2{}_{RPT}(\theta_1) - \cot^2{}_{RPT}(\theta_1) = 1$

Proof: Consider $\mathrm{Cosec}^2{}_{RPT}(\theta_1) - \cot^2{}_{RPT}(\theta_1) = \left(\frac{a}{h}\right)^2 - \left(\frac{a}{b}\right)^2 = a^2\left(\left(\frac{1}{h}\right)^2 - \left(\frac{1}{b}\right)^2\right) = a^2 * \frac{1}{a^2} = 1$. We can easily verify the result by taking some reciprocal Triple.

Lemma K: $\mathrm{Cosec}^2{}_{RPT}(\theta_2) - \cot^2{}_{RPT}(\theta_2) = 1$

Proof: Consider $\mathrm{Cosec}^2{}_{RPT}(\theta_2) - \cot^2{}_{RPT}(\theta_2) = \left(\frac{b}{h}\right)^2 - \left(\frac{b}{a}\right)^2 = b^2\left(\left(\frac{1}{h}\right)^2 - \left(\frac{1}{a}\right)^2\right) = b^2 * \frac{1}{b^2} = 1$. We can easily verify the result by taking some reciprocal Triple.

Lemma L: $\mathrm{Sec}^2{}_{RPT}(\theta_1) - \tan^2{}_{RPT}(\theta_1) = 1$

Proof: Consider $\mathrm{Sec}^2{}_{RPT}(\theta_1) - \mathrm{Tan}^2{}_{RPT}(\theta_1) = \left(\frac{b}{h}\right)^2 - \left(\frac{b}{a}\right)^2 = b^2\left(\left(\frac{1}{h}\right)^2 - \left(\frac{1}{a}\right)^2\right) = b^2 * \frac{1}{b^2} = 1$. We can easily verify the result by taking some reciprocal Triple.

Lemma M: $\mathrm{Sec}^2{}_{RPT}(\theta_2) - \tan^2{}_{RPT}(\theta_2) = 1$

Proof: Consider $\mathrm{Sec}^2{}_{RPT}(\theta_2) - \tan^2{}_{RPT}(\theta_2) = \left(\frac{a}{h}\right)^2 - \left(\frac{a}{b}\right)^2 = a^2\left(\left(\frac{1}{h}\right)^2 - \left(\frac{1}{b}\right)^2\right) = a^2 * \frac{1}{a^2} = 1$, We can easily verify the result by taking some reciprocal Triple.

Lemma N: $\sin_{RPT}(\theta_1 + \theta_2) = 1$

Proof:$\sin_{RPT}(\theta_1 + \theta_2)=\sin_{RPT}(\theta_1)\cos_{RPT}(\theta_2) + \cos_{RPT}(\theta_1)\sin_{RPT}(\theta_2) = \frac{h}{a} * \frac{h}{a} + \frac{h}{b} * \frac{h}{b} = h^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right)$
$= h^2 * \frac{1}{h^2} = 1$. It will clearly indicate that $\theta_1 + \theta_2 = \frac{\pi}{2}$

Lemma O: $\mathrm{Cos}_{RPT}(\theta_1 + \theta_2) = 0$

Proof: $\mathrm{Cos}_{RPT}(\theta_1 + \theta_2)=\cos_{RPT}(\theta_1)\cos_{RPT}(\theta_2) - \sin_{RPT}(\theta_1)\sin_{RPT}(\theta_2)= \frac{h}{b} * \frac{h}{a} - \frac{h}{a} * \frac{h}{b} = 0$.

It will clearly indicate that $\theta_1 + \theta_2 = \frac{\pi}{2}$.

In this way to can go to prove all properties of Trigonometric Functions for Reciprocal Pythagorean Triples also applicable for Reciprocal Pythagorean Triples. It is very useful to find Area of Triangle, Orthocenter of a triangle.

CONCLUSION

In this paper, focused to study Algebraic Structure and some Homomorphism functions on Set of Pythagorean Triples$P_T = \{(x, y, z)\epsilon Z^3 : x^2 + y^2 = z^2\}$ and Set of Reciprocal Pythagorean Triples. $RP_T = \left\{(x, y, z)\epsilon Z^3 : \frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{z^2}\right\}$. With respect to following binary operation of Lemma A& Lemma B. For some $P_1 = (x_1, y_1, z_1)\epsilon P_T, P_2 = (x_2, y_2, z_2)\epsilon P_T$with $p_1 . p_2 = \left\{\begin{matrix}(|y_1y_2 - x_1x_2|, x_1y_2 + x_2y_1, z_1z_2) \ldots [A] \\ (x_1x_2, y_1z_2 + y_2z_1, y_1y_2 + z_1z_2) \ldots [B]\end{matrix}\right\}$. Also If (x , y , z) ϵ RP_T Then (x z , y z , x y) ϵ P_T and vice versa. It follows, Also we are introduce to study under the above binary operation, following Homomorphism functions on Reciprocal Pythagorean triples are $Rpt_o(x)= \left\{\left(\frac{x(x^2+1)}{2}, \frac{(x^2-1)(x^2+1)}{4}, \frac{x(x^2-1)}{2}\right)\right\}$,x is an odd number greater than 1

$Rpt_e(x)=\left\{\left(x\left(\left(\frac{x}{2}\right)^2 + 1\right)\right), \left(\left(\frac{x}{2}\right)^2 - 1\right)\left(\left(\frac{x}{2}\right)^2 + 1\right), x\left(\left(\frac{x}{2}\right)^2 - 1\right)\right\}$, x is an even number greater than 2 and $Rpt(p, q)= \{(p^2 + q^2)(p^2 - q^2),\ 2pq(p^2 +$

$q^2), 2pq(p^2q^2)$: for some integers $p > q\}$. Also Introduce to study, some other inherent properties of Reciprocal Pythagorean Triples.

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