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The Cyclic Edge Connectivity Number of an Arithmetic Graph $G=V_n$

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Abstract: **The edge set F is a cyclic edge cut if $G-F$ is disconnected and at least two of its components contain cycle. The minimum cardinality of cyclic edge cut is called cyclic edge connectivity number and it is denoted by $\lambda_c(G)$. In this article, the cyclic edge connectivity of an arithmetic graph is studied. We categorized arithmetic graphs which are cyclically separable and not cyclically separable. Also, it is shown that for all arithmetic graphs other than $G=V_n$ where $n = p_1^{a_1} \times p_2^{a_2}$; $a_1 \geq 1$ and $a_2=1$ or $a_1=a_2=2$ is cyclically separable. Moreover, all arithmetic graphs other than $G=V_n$, $n = p_1^{a_1} \times p_2^{a_2}$; $a_1>3$ & $a_2 \geq 2$ are cyclically optimal.**

Index Terms: **arithmetic graph, cyclic edge cut, cyclic edge connectivity number, cyclically optimal.**

I. Introduction

Graph theory terminology and notation not given here; we follow [2]. The concept of cyclic edge connectivity was introduced by Tiat in the proof of the four colour theorem [11]. The authors Haining Jiang, Jixiang Meng, Yingzhi Tian studied cyclic edge connectivity of Half vertex transitive graphs in [4]. The definition of an arithmetic graph is studied from [12]. [8]The arithmetic graph V n is defined as a graph with its vertex set is the set consists of the divisors of n (excluding 1) where n is a positive integer and $n = p_1^{a_1} \times p_2^{a_2} \times \ldots \ldots \times p_r^{a_r}$ where pi's are distinct primes and ai's≥1 and two distinct vertices a, b which are not of the same parity are adjacent in this graph if (a, b)=p_i, for some i , 1≤i ≤ r. The vertices a and b are said to be of the same parity if both a and b are the powers of the same prime, for instance $a = p^3$, $b = p^4$. In [5] the connectivity number of an arithmetic graph is studied by L. Mary Jenitha and S. Sujitha. Later the average connectivity of the same graph is discussed by the same authors in [6]. As well as the concepts of connectivity of complement of an arithmetic graph is also studied by same authors in [7]. Also, various authors studied different parameters of an arithmetic graph. In this paper we investigated the cyclic edge connectivity concepts for an arithmetic graph $G = V_n$. The following theorems are used in sequel.

Definition 1.1. [4] The edge set F is a cyclic edge cut if $G-F$ is disconnected and at least two of its components contains cycle. The minimum cardinality of cyclic edge cut is called cyclic edge connectivity number and it is denoted by $\lambda_c(G)$.

Theorem 1.2. [10] For an arithmetic graph $G = V_n$, $n = p_1^{a_1} \times p_2^{a_2} \times \ldots \ldots \times p_r^{a_r}$ then the number of vertices *of G is* $|V| = \prod_{i=1}^{r}(a_i+1)-1$.

Theorem 1.3. [6] For an arithmetic graph $G = V_n$, $n = p_1^{a_1} \times p_2^{a_2}$ where p_1 and p_2 are distinct primes, a_1, $a_2 \geq 1$ then $\mathcal{E} = 4a_1a_2-a_1-a_2$, where $\mathcal{E}$ is the size of the graph G.

Theorem 1.4. [6] For an arithmetic graph $G = V_n$, $n = p_1^{a_1} \times p_2^{a_2}$ where p_1 and p_2 are distinct primes, a_1, $a_2 \geq 1$ then G is a bipartite graph.

Theorem 1.5. [6] Let $G=V_n$ be an arithmetic graph $n = p_1^{a_1} \times p_2^{a_2} \times \ldots \ldots \times p_r^{a_r}$ for any vertex $u = \prod_{i\in B} p_i^{\alpha_i}$ where $B \subseteq \{1,2,\ldots,r\}$,$1 \leq \alpha_i \leq a_i \forall i \in B$

(1) If u=p_j where $j \in \{1,2,\ldots,r\}$, then

$\deg(u) = \left[a_j \prod_{i=1, i\neq j}^{r}(a_i+1)-1\right] - |a_j-1|$.

(2) If $u = p_i^{\alpha_i}$ $1 \leq \alpha_i \leq a_i \forall i \in B$, then

$\deg(u) = \left[\prod_{i=1, i\notin B}^{r}(a_i+1)\right]-1$.

(3) If $u = \prod_{i\in B} p_i^{\alpha_i}$, $|B| \geq 2$, $1 < \alpha_i \leq a_i \forall i \in B$ then $\deg(u) = |B| \prod_{i=1, i\notin B}^{r}(a_i+1)$.

(4) If $u = \prod_{i\in B} p_i^{\alpha_i}$, $\alpha_i = 1$ for some $i \in B' \subseteq B$, then

$\deg(u) = [|B-B'| + \sum_{i\in B'} a_i] \prod_{i=1, i\notin B}^{r}(a_i+1)$ where B is the number of prime products in u, B' is the number of primes

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having power 1 in the chosen vertex u.

Theorem 1.6. [8] Let $G=V_n$ be an arithmetic graph $n = {p_1}^{a_1} \times {p_2}^{a_2} \times \ldots\ldots \times {p_r}^{a_r}, r > 2$ such that at least one of a_i, $i \in \{1,2,\ldots,r\}$ does not equal one. Then $\Delta(G) = \left[a_j \prod_{i=1,i\neq j}^{r}(a_i + 1) - 1\right] - \left|a_j - 1\right|$ where a_j is the maximum exponent of $p_i, i \in \{1,2,\ldots,r\}$, $\delta(G) = r$.

Theorem 1.7. [10] Let $G=V_n$ be an arithmetic graph $n = {p_1}^{a_1} \times {p_2}^{a_2} \times \ldots\ldots \times {p_r}^{a_r}, r > 2$ such that at least one of a_i, ,$i \in \{1,2,\ldots,r\}$ does not equal one. Then

(1) $\Delta(G) = \left[a_j \prod_{i=1,i\neq j}^{r}(a_i + 1) - 1\right]$ where a_j is the maximum exponent of $p_i, i \in \{1,2,\ldots,r\}$. (2) $\delta(G) = r$.

In the study of a cyclic edge connectivity of an arithmetic graph $G = V_n$, we found that for every $G = V_n$ other than $n = p_1 \times p_2$ have cycles. Also, the number of cycles in $G = V_n$ depends on the number of primes and prime powers in n. If the number of primes and prime power increases the number of cycles also increases. As we have more cycles in $G = V_n$ we need to choose a cycle with $\sum_{v_i \in C} d(v_i)$ is minimum. The required cycle can be chosen in a particular way, which is discussed in the corresponding theorems.

II. Infinite Cyclic Edge Connectivity Of An Arithmetic Graph

In this section, we identify the graphs which are not cyclically edge separable and their cyclic edge connectivity number. We observe that the cyclic edge connectivity number $\lambda_c(G) = \infty$, if the number of primes in n doesn't exceed two.

Theorem 2.1. For an arithmetic graph $G=V_n$, $n = {p_1}^{a_1} \times {p_2}^{a_2}$ for $a_1 \geq 1$ has no cyclic edge cut. The cyclic edge connectivity number $\lambda_c(G) = \infty$.

Proof.

Case (i) If $a_1 = 1$ then the arithmetic graph $G = V_n$, $n = p_1 \times p_2$ is a tree. Hence $\lambda_c(G) = \infty$.

Case (ii) If $a_1 > 1$ then the vertex set of G be $V(G) = \{p_1, {p_1}^2, \ldots\ldots, {p_1}^{a_1}, p_2, p_1 \times p_2, {p_1}^2 \times p_2, \ldots\ldots, {p_1}^{a_1} \times p_2\}$ and by Theorem 1.4, G is a bipartite graph whose partition $A=\{p_1, {p_1}^2, \ldots\ldots, {p_1}^{a_1}, p_2\}$ and $B=\{p_1 \times p_2, {p_1}^2 \times p_2, \ldots\ldots, {p_1}^{a_1} \times p_2\}$. Since G is a bipartite there is no cycle of length 3. Also, in the first partite all vertices other than p_1 and p_2 are pendant vertices. So, there is no two vertex disjoint cycles. Hence there is no cyclic edge cut. Hence $\lambda_c(G) = \infty$.

Theorem 2.2. Let $G=V_n$, be an arithmetic graph $n = {p_1}^2 \times {p_2}^2$ then the cyclic edge connectivity number $\lambda_c(G) = \infty$.

Proof.

By Theorem 1.4, G is a bipartite graph whose partitions $A=\{p_1, {p_1}^2, p_2, {p_2}^2\}$ and $B= \{p_1 \times p_2, {p_1}^2 \times p_2, p_1 \times {p_2}^2, {p_1}^2 \times {p_2}^2\}$ also by the definition of arithmetic graph there is no disjoint cycles. Hence $\lambda_c(G) = \infty$.

III. Finite Cyclic Edge Connectivity Of An Arithmetic Graph

Here, we discuss the arithmetic graphs which are cyclically edge separable and the cyclic edge connectivity number for the same.

Theorem 3.1. The cyclic edge connectivity number $\lambda_c(G) = 5$ if $G=V_n$, is an arithmetic graph where $n = {p_1}^{a_1} \times {p_2}^{a_2}$, $a_1 \geq 3, a_2 = 2$.

Proof. Let $G = V_n$ be an arithmetic graph where $n = {p_1}^{a_1} \times {p_2}^{a_2}$ with $a_1 \geq 3, a_2 = 2$, then the vertex set of G be $V(G)=\{p_1, {p_1}^2, \ldots\ldots, {p_1}^{a_1}, p_2, {p_2}^2, p_1 \times p_2, {p_1}^2 \times p_2, \ldots\ldots, {p_1}^{a_1} \times p_2, p_1 \times {p_2}^2, {p_1}^2 \times {p_2}^2, \ldots, {p_1}^{a_1} \times {p_2}^2\}$ and by Theorem 1.4, G is a bipartite graph with partition $A=\{p_1, {p_1}^2, \ldots\ldots, {p_1}^{a_1}, p_2, {p_2}^2\}$ and $B=\{p_1 \times p_2, {p_1}^2 \times p_2, \ldots\ldots, {p_1}^{a_1} \times p_2, p_1 \times {p_2}^2, {p_1}^2 \times {p_2}^2, \ldots, {p_1}^{a_1} \times {p_2}^2\}$. Let us choose all minimum degree vertices in partition A which is of the form ${p_1}^m$ where $2 \leq m \leq a_1$ and its neighbor vertices in the partition B which is $p_1 \times p_2$ and $p_1 \times {p_2}^2$. Remove the edges which are incident on $p_1 \times p_2$ from $p_1, p_2, {p_2}^2$ and remove the edges which are incident on $p_1 \times {p_2}^2$ from p_1 and p_2. Thus, the removal of five edges results the graph disconnected with exactly two components G_1 and G_2 where G_1 contains $\binom{a_1-1}{2}$ cycles of length 4, and G_2 is a connected component contains cycles. Since the removal of less than five edges disconnect the graph but do not satisfy the definition of cyclic edge cut, $\lambda_c(G) = 5$.

Theorem 3.2. Let $G=V_n$ be an arithmetic graph, $n = {p_1}^{a_1} \times {p_2}^{a_2}$ where $a_1 \geq a_2 > 2$ then the cyclic edge connectivity number $\lambda_c(G) = 2a_2 + a_1 - 3$.

Proof. Let $G = V_n$ be an arithmetic graph where $n = {p_1}^{a_1} \times {p_2}^{a_2}$ with $a_1 \geq a_2 > 2$. The vertex set of G is $V(G)=\{p_1, {p_1}^2, \ldots\ldots, {p_1}^{a_1}, p_2, {p_2}^2, \ldots\ldots, {p_2}^{a_2}, p_1 \times p_2, \ldots\ldots, p_1 \times {p_2}^{a_2}, {p_1}^2 \times p_2, \ldots\ldots, {p_1}^2 \times {p_2}^{a_2}, \ldots\ldots, {p_1}^{a_1} \times p_2, \ldots, {p_1}^{a_1} \times {p_2}^{a_2}\}$. By Theorem 1.4, G is bipartite graph with partitions, A and B where $A=\{p_1, {p_1}^2, \ldots\ldots, {p_1}^{a_1}, p_2, {p_2}^2, \ldots\ldots, {p_2}^{a_2}\}$ and $B = \{p_1 \times p_2, \ldots\ldots, p_1 \times {p_2}^{a_2}, {p_1}^2 \times p_2, \ldots\ldots, {p_1}^2 \times {p_2}^{a_2}, \ldots\ldots, {p_1}^{a_1} \times p_2, \ldots, {p_1}^{a_1} \times {p_2}^{a_2}\}$. Since $a_1 \geq a_2$ then $d({p_1}^m) \leq d({p_2}^n)$ for $1 < m \leq a_1$, $1 < n \leq a_2$. Let $A_1 \subset A$ be the set of minimum degree vertices and $B_1 \subset B$ be the set of vertices adjacent to A_1. Let F_1 be the set of edges which are incident on $p_1 \times p_2$ from A_1. Therefore $|F_1| = |A_1| = a_1 - 1$ edges and F_2 be the set of edges which incident on $\{p_1 \times {p_2}^2, p_1 \times {p_2}^3, \ldots\ldots, p_1 \times {p_2}^{a_2}\}$ from p_1 and p_2 (i.e) $|F_2| = 2|B_1| = 2(a_2 - 1)$. So, if we remove $|F| = |F_1| + |F_2| = 2a_2 + a_1 - 3$ edges from the graph G, the resultant graph is a disconnected graph with two components G_1 and G_2 where G_1 has a cycle of length $2a_2$. Also, every pair of vertices in G_1 has at least one path hence G_1 is connected. Also, by the definition of arithmetic graph, two vertices say $p_1, p_2 \in V(G_2)$ is adjacent to every vertices of the form $\{{p_1}^m \times {p_2}^n; 1 \leq m \leq a_1, 1 \leq n \leq a_2\}$ and the vertices of the form $\{{p_2}^n; 1 \leq$

$n \le a_2\}$ are adjacent to the vertex $p_1 \times p_2$. Hence G_2 is a connected graph. Also, we can easily observe that G_2 contains a cycle.

To prove F is minimum, since we have chosen the minimum degree vertices from partition A and all their neighbors other than $p_1 \times p_2$ are included in the cycle. The cyclic edge cut is minimum. Hence, we have $\lambda_c(G) = |F| = 2a_2 + a_1 - 3$.

Theorem 3.3. For an arithmetic graph $G=V_n$, $n = p_1 \times p_2 \times \ldots \times p_r$, $r \ge 3$ then the cyclic edge connectivity number.

$(i)\ \lambda_c(G) = 2^{(r-1)} + 2\left(\frac{r+1}{2}\right)2^{\left(\frac{r-1}{2}\right)} - 7$ if r is odd.

$$(ii)\ \lambda_c(G) = 2^{(r-1)} + \left(\frac{r}{2}\right)2^{\left(\frac{r}{2}\right)} + \left(\frac{r}{2}+1\right)2^{\left(\frac{r}{2}-1\right)} - 7 \text{ if r is even}$$

Proof. By Theorem 1.2, $|V|=2^r - 1$, here we have two cases,

Case(i) r is odd and $r \ge 3$

Consider the cycle C_3 in which $\sum_{v_i \in C} d(v_i)$ is minimum, C_3 can be chosen in the following way (1) One vertex must be a single prime let it be p_i. Since, $n = p_1 \times p_2 \times \ldots \times p_{\frac{r}{2}} \times p_{\frac{r+1}{2}} \times p_{\left(\frac{r+1}{2}+1\right)} \times \ldots \times p_r$ and by Theorem1.6, with more prime products has less degree the second vertex must be the product of $\frac{r+1}{2}$ primes including p_i.

The third vertex must be a product of remaining $\frac{r-1}{2}$ primes and p_i, otherwise either it violates the adjacency or minimum degree. Since v_i and v_j are adjacent in $G=V_n$

if the number of primes in v_i and v_j is less than or equal to $r+1$ and $\gcd(v_i, v_j) = p_i$. In this way we choose a cycle $V(C_3) = \left\{p_1, p_1 \times p_2 \times \ldots \times p_{\frac{r}{2}} \times p_{\frac{r+1}{2}}, p_1 \times p_{\left(\frac{r+1}{2}+1\right)} \times \ldots \times p_r\right\}$. Now let F be the set of edges incident on the vertices of C_3 other than the edges of the cycle C_3. The induced graph of *V(G)-F* is a disconnected graph with two components each contains cycles. Thus, it satisfies the definition of cyclic edge connectivity, hence the set of edges in F is called the cyclic edge cut, since the cycle which we have chosen is $\sum_{v_i \in C} d(v_i)$ is minimum in G. The cardinality of F is called the cyclic edge connectivity number $\lambda_c(G)$.

Now $\lambda_c(G) = d(p_1) - 2 + d(p_1 \times p_2 \times \ldots \times p_{\frac{r}{2}} \times p_{\frac{r+1}{2}}) - 2 + d\left(p_1 \times p_{\left(\frac{r+1}{2}+1\right)} \times \ldots \times p_r\right) - 2$. By Theorem 1.5, we have $\lambda_c(G) = 2^{(r-1)} + 2\left(\frac{r+1}{2}\right)2^{\left(\frac{r-1}{2}\right)} - 7$.

Case(ii) Assume that r is even and $r \ge 3$ similar as case (i) choose a cycle C_3 such that the sum of the degree is minimum, let the vertices of the cycle be $V(C_3) = \left\{p_1, p_1 \times p_2 \times \ldots \times p_{\frac{r}{2}}, p_1 \times p_{\left(\frac{r}{2}+1\right)} \times \ldots \times p_r\right\}$. Here p_1 is the only prime which is common to all the three vertices of C_3. Similar as case (i) here also we get

$\lambda_c(G) = d(p_1{}^{a_1}) - 2 + d(p_1 \times p_2 \times \ldots \times p_{\frac{r}{2}}) - 2 + d\left(p_1 \times p_{\left(\frac{r}{2}+1\right)} \times \ldots \times p_r\right) - 2$. By Theorem 1.5, we have $\lambda_c(G) = 2^{(r-1)} + \left(\frac{r}{2}\right)2^{\left(\frac{r}{2}\right)} + \left(\frac{r}{2}+1\right)2^{\left(\frac{r}{2}-1\right)} - 7$.

Theorem 3.4. For any arithmetic graph $G=V_n$, $n = p_1{}^{a_1} \times p_2{}^{a_2} \times \ldots \times p_r{}^{a_r}, r \ge 3$, $a_i > 1$ for exactly one i the cyclic edge connectivity number

$\lambda_c(G) = 2^{(r-1)} + 2\left(a_1 + \frac{r-1}{2}\right)2^{\left(\frac{r-1}{2}\right)} - 7$ (i) if r is odd

(ii) $\lambda_c(G) = 2^{(r-1)} + \left(a_1 + \frac{r-2}{2}\right)2^{\left(\frac{r}{2}\right)} + \left(a_1 + \frac{r}{2}\right)2^{\left(\frac{r}{2}-1\right)} - 7$ if r is even

Proof.

Case(i) r is odd

Let $G=V_n$ be an arithmetic graph where $n = p_1{}^{a_1} \times p_2{}^{a_2} \times \ldots \times p_r{}^{a_r}$, $a_i \ne 1$ for exactly one i, then the vertex set $V(G) = \{p_1, p_1{}^2, \ldots, p_1{}^{a_1}, p_2, p_3, \ldots, p_r, p_1 \times p_2, \ldots p_1{}^{a_1} \times p_r, \ldots, p_1 \times p_2 \times p_3, \ldots, p_1 \times p_2 \times \ldots \times p_r, \ldots, p_1{}^{a_1} \times p_2 \times \ldots \times p_r\}$. Let us arrange the primes of n in such a way that the greatest power prime in the first position. Consider the vertices of the cycle $V(C_3) = \left\{p_1{}^{a_1}, p_1 \times p_2 \times \ldots \times p_{\frac{r}{2}} \times p_{\frac{r+1}{2}}, p_1 \times p_{\left(\frac{r+1}{2}+1\right)} \times \ldots \times p_r\right\}$. (i) we choose the first vertex of the cycle as $p_1{}^{a_1}, a_1 \ne 1$ since the prime vertex p_1 has the greatest degree among the primes and prime power vertices of G.

(ii) second vertex consist of first $\frac{(r+1)}{2}$ primes including p_1, Since the vertices with more prime products will have less degree.

(iii) third vertex must be the product of $r - \frac{r+1}{2}$ primes and p_1, primes other than p_1 should not be in the second vertex, otherwise it violates the adjacency of second and third vertex. Let F be the set of all edges incident on the vertices of C_3 other than the edges of the cycle C_3. The induced graph of *V(G)-F* is a disconnected graph with two components each contains a cycle, thus it satisfies the definition of cyclic edge cut. Also, $\sum_{v_i \in C} d(v_i)$ is minimum hence the cyclic edge connectivity number is $|F|$.

Now $\lambda_c(G) = d(p_1{}^{a_1}) - 2 + d(p_1 \times p_2 \times \ldots \times p_{\frac{r}{2}} \times p_{\frac{r+1}{2}}) - 2 + d\left(p_1 \times p_{\left(\frac{r+1}{2}+1\right)} \times \ldots \times p_r\right) - 2$.

By Theorem 1.5, we have $\lambda_c(G) = [\prod_{i=1, i \notin B}^{r}(a_i + 1)] - 1 - 2 + [|B - B'| + \sum_{i \in B'} a_i]\ \prod_{i=1, i \notin B}^{r}(a_i + 1) - 2 + [|B - B'| + \sum_{i \in B'} a_i] \prod_{i=1, i \notin B}^{r}(a_i + 1) - 2$.

$= 2^{(r-1)} + 2\left(a_1 + \frac{r-1}{2}\right)2^{\left(\frac{r-1}{2}\right)} - 7$.

Case(ii) r is even

Let $G=V_n$ be an arithmetic graph where $n = p_1{}^{a_1} \times p_2{}^{a_2} \times \ldots \times p_r{}^{a_r}$, $a_i \ne 1$ for exactly one i, r is even and $r \ge 3$, then the vertex set $V(G) = \{p_1, p_1{}^2, \ldots, p_1{}^{a_1}, p_2, p_3, \ldots, p_r, p_1 \times$

$p_2, \ldots\ldots p_1{}^{a_1} \times p_r, \ldots\ldots, p_1 \times p_2 \times p_3, \ldots\ldots, p_1 \times p_2 \times \ldots\ldots \times p_r, \ldots\ldots, p_1{}^{a_1} \times p_2 \times \ldots \times p_r\}$. Let us arrange the primes of n in such a way that the power of p_1 is greater than 1. Consider the vertices of the cycle C_3 whose $V(C_3) = \{p_1{}^{a_1}, p_1 \times p_2 \times \ldots\ldots \times p_{\frac{r}{2}}, p_1 \times p_{\left(\frac{r}{2}+1\right)} \times \ldots. \times p_r\}$. Similar as case(i) we get

$\lambda_c(G) = d(p_1{}^{a_1}) - 2 + d(p_1 \times p_2 \times \ldots\ldots \times p_{\frac{r}{2}}) - 2 + d\left(p_1 \times p_{\left(\frac{r}{2}+1\right)} \times \ldots. \times p_r\right) - 2.$

$= 2^{(r-1)} + \left(a_1 + \frac{r-2}{2}\right) 2^{\left(\frac{r}{2}\right)} + \left(a_1 + \frac{r}{2}\right) 2^{\left(\frac{r}{2}-1\right)} - 7$.

Example 3.5. Consider an arithmetic graph $G = V_{72}$ where $72 = 2^3 \times 3^2$ given in Figure1. Here we choose one of the cycle of length 4 whose degree sum is minimum as $C_4 : 2^3 \rightarrow 2 \times 3^2 \rightarrow 2^2 \rightarrow 2 \times 3 \rightarrow 2^3$ and by the removal of the set of edges $F = \{2 \times 3\ 2, 2 \times 3\ 3, 2 \times 3\ 3^2, 2 \times 3^2 3, 2 \times 3^2 2\}$ the resultant graph is a disconnected graph with two components G_1, G_2 where G_1 is a cycle C_4 and G_2 is a connected graph containing cycles. Hence

$\lambda_c(G) = d(2^2) + d(2^3) + d(2 \times 3) + d(2 \times 3^2) - 8 = 5.$

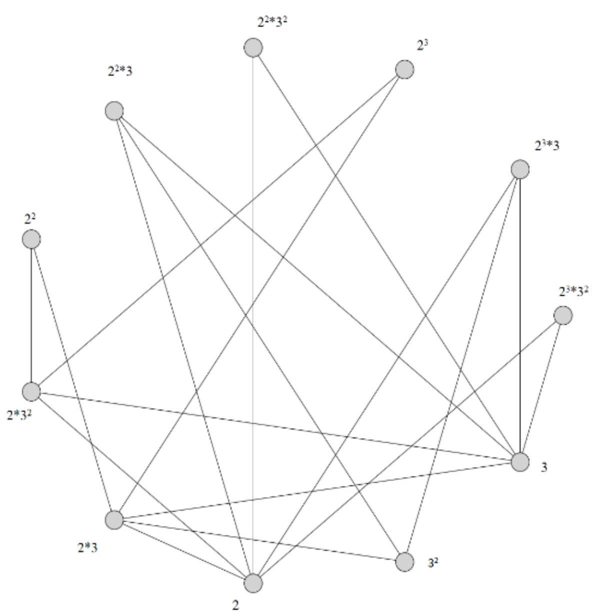

Figure 1: Arithmetic Graph G=V 72

Theorem 3.6 If $G = V_n$ is an arithmetic graph where $n = p_1{}^{a_1} \times p_2{}^{a_2} \times \ldots\ldots \times p_r{}^{a_r}, r \geq 3$, $if\ a_1 \geq a_2 > 1$ then the cyclic edge connectivity number

$\lambda_c(G) = (a_2+1)2^{(r-2)} + 2\left(a_1 + \frac{r-1}{2}\right)\left[2^{\left(\frac{r-1}{2}\right)} + (a_2+1)2^{\left(\frac{r-3}{2}\right)}\right] - 7$ if r is odd

$\lambda_c(G) = (a_2+1)2^{(r-2)} + (r+2a_1)2^{\left(\frac{r}{2}-3\right)} + (2a_1 + r - 2)\ 2^{\left(\frac{r}{2}-1\right)} - 7$ if r is even

Proof. Let $G = V_n$ be an arithmetic graph where $n = p_1{}^{a_1} \times p_2{}^{a_2} \times \ldots. \times p_r{}^{a_r}$ if $a_1 \geq a_2 > 1$ and $r \geq 3$. The vertex set of G be $V(G) = \{p_1, p_1{}^2, \ldots., p_1{}^{a_1}, p_2, p_2{}^2, \ldots., p_2{}^{a_2}, p_3, \ldots., p_r, p_1 \times p_2, \ldots\ldots, p_1{}^{a_1} \times p_r, p_2 \times p_r, p_2{}^2 \times p_3, \ldots\ldots, p_2{}^2 \times p_r, \ldots, p_1 \times p_2 \times p_3, \ldots\ldots, p_1 \times p_2 \times \ldots\ldots \times p_r, \ldots\ldots, p_1{}^{a_1} \times p_2 \times \ldots \times p_r, \ldots\ldots, p_1 \times p_2{}^{a_2} \times p_3 \times \ldots \times p_r, p_1{}^{a_1} \times p_2{}^{a_2} \times p_3 \times \ldots \times p_r, \}$. Let us arrange p_i in such a way that $a_1 \geq a_2 > 1$.

Case(i) If r is odd, then we choose cycle C_3 whose vertex set is $V(C_3) = \left\{p_1{}^{a_1}, p_1 \times p_2{}^{a_2} \times \ldots\ldots \times p_{\frac{r}{2}} \times p_{\frac{r+1}{2}}, p_1 \times p_{\left(\frac{r+1}{2}+1\right)} \times \ldots. \times p_r\right\}$

(i) we choose first vertex of the cycle as $p_1{}^{a_1}$, $a_1 \neq 1$, since the prime vertex p_i, $i \in \{1,2,..,r\}$ has the greatest degree among the prime and prime power vertices of G.

(ii)second vertex consists of first $\frac{r+1}{2}$ primes including $p_2{}^{a_2}$, since the vertices with more prime products will have less degree.

(iii)Third vertex must be the product of r-$\frac{r+1}{2}$ primes and p_1. Also, the primes in the third vertex other than p_1 is not in the second vertex. Otherwise, it violates the adjacency of second and third vertex. Similar as above proof we get $\lambda_c(G) = d(p_1{}^{a_1}) - 2 + d(p_1 \times p_2{}^{a_2} \times \ldots\ldots \times p_{\frac{r+1}{2}}) - 2 + d\left(p_1 \times p_{\left(\frac{r+1}{2}+1\right)} \times \ldots. \times p_r\right) - 2.$

$$= [\prod_{i=1, i\notin B}^{r} (a_i+1)] - 1 - 2 + [|B - B'| + \sum_{i \in B'} a_i] \prod_{i=1, i\notin B}^{r} (a_i+1) - 2 + [|B - B'| + \sum_{i \in B'} a_i] \prod_{i=1, i\notin B}^{r} (a_i+1) - 2.$$

$$= (a_2+1)2^{r-2} - 1 + \left[1 + a_1 + \frac{r-3}{2}\right] 2^{\frac{r-1}{2}} + \left[a_1 + \frac{r-1}{2}\right][a_2+1]2^{\frac{r-3}{2}} - 6.$$

$$= (a_2+1)2^{(r-2)} + 2\left(a_1 + \frac{r-1}{2}\right)\left[2^{\left(\frac{r-1}{2}\right)} + (a_2+1)2^{\left(\frac{r-3}{2}\right)}\right] - 7\ .$$

Case(ii) If r is even, then we choose the cycle C_3 as $V(C_3) = \{p_1{}^{a_1}, p_1 \times p_2{}^{a_2} \times \ldots\ldots \times p_{\frac{r}{2}}, p_1 \times p_{\left(\frac{r}{2}+1\right)} \times \ldots. \times p_r\}$. Similar as case(i) we get $\lambda_c(G) = d(p_1{}^{a_1}) - 2 + d(p_1 \times p_2{}^{a_2} \times \ldots\ldots \times p_{\frac{r}{2}}) - 2 + d\left(p_1 \times p_{\left(\frac{r}{2}+1\right)} \times \ldots. \times p_r\right) - 2.$

$= (a_2+1)2^{(r-2)} - 3 + \left[|1| + a_1 + \frac{r-4}{2}\right] 2^{\left(\frac{r}{2}\right)} - 2 + \left(a_1 + \frac{r}{2}\right)[a_2+1]2^{\left(\frac{r}{2}-2\right)} - 2.$

$$= (a_2+1)2^{(r-2)} + (r+2a_1)2^{\left(\frac{r}{2}-3\right)} + (2a_1 + r - 2)2^{\left(\frac{r}{2}-1\right)} - 7.$$

Theorem 3.7. Let $G=V_n$ be an arithmetic graph, $n = p_1^{a_1} \times p_2^{a_2} \times \dots\dots \times p_r^{a_r}, r \geq 3, if\ a_i \neq 1$ for $i \in \{1,2,3\}$ then the cyclic edge connectivity number

(i) $\lambda_c(G) = (a_2+1)(a_3+1)2^{(r-3)} + [2a_1 + r - 1]\left[2^{\frac{(r-5)}{2}}\right](a_2+1)(a_3+1) - 7$ if r is odd

(ii) $\lambda_c(G) = (a_2+1)(a_3+1)2^{(r-3)} + [2a_1 + r - 2]2^{(\frac{r}{2}-1)} + (2a_1 + r)2^{(\frac{r}{2}-2)} - 7$ if r is even

Proof.

Case(i) r is odd and r ≥ *3*

Let $G=V_n$ be an arithmetic graph where $n = p_1^{a_1} \times p_2^{a_2} \times \dots\dots \times p_r^{a_r}, r \geq 3$ if $a_i \neq 1$ for $i \in \{1,2,3\}$ then the vertex set *V(G)*= $\{p_1, p_1^2, \dots, p_1^{a_1}, p_2, p_2^2, \dots\dots, p_2^{a_2},$ $p_3, p_3^2, \dots, p_3^{a_3} \dots, p_r, p_1 \times p_2, \dots\dots, p_1^{a_1} \times p_r, p_2 \times p_3, p_2^2 \times p_3, \dots\dots, p_2^2 \times p_r, \dots, p_1 \times p_2 \times p_3, \dots\dots, p_1 \times p_2 \times \dots\dots \times p_r, \dots\dots, p_1^{a_1} \times p_2 \times \dots \times p_r, , \dots\dots, p_1 \times p_2^{a_2} \times p_3 \times \dots \times p_r, p_1^{a_1} \times p_2^{a_2} \times p_3 \times \dots \times p_r, \dots, p_1^{a_1} \times p_2^{a_2} \times p_3^{a_3} \times \dots \times p_r\}$. Let us arrange *n* in such a way that the power of p_1 and p_2 are greater than one and $a_1 \geq a_2$. Also, the prime which is of having power greater than one will be arranged in the $(\frac{r+1}{2}+1)$ th place of *n.*

The vertices of the cycle $V(C_3) = \left\{p_1^{a_1}, p_1 \times p_2^{a_2} \times \dots\dots \times p_{\frac{r}{2}} \times p_{\frac{r+1}{2}}, p_1 \times p_{(\frac{r+1}{2}+1)}^{a_{(\frac{r+1}{2}+1)}} \times \dots\dots \times p_r\right\}$ Similar to Theorem 3.6, we get

$\lambda_c(G) = d(p_1^{a_1}) - 2 + d(p_1 \times p_2^{a_2} \times \dots\dots \times p_{\frac{r+1}{2}}) - 2 + d\left(p_1 \times p_{(\frac{r+1}{2}+1)}^{a_{(\frac{r+1}{2}+1)}} \times \dots\dots \times p_r\right) - 2.$

$= (a_2+1)(a_3+1)2^{(r-3)} + [2a_1 + r - 1][2^{\left(\frac{r-5}{2}\right)}](a_2+1)(a_3+1) - 7.$

Case(ii) *r* is even and *r* ≥ 3

Similar as case (i) the vertices of cycle is $V(C_3)=\{ p_1^{a_1}, p_1 \times p_2^{a_2} \times \dots\dots \times p_{\frac{r}{2}}, p_1 \times p_{(\frac{r}{2}+1)}^{a_{(\frac{r}{2}+1)}} \times \dots\dots \times p_r \}$. We get

$\lambda_c(G) = d(p_1^{a_1}) - 2 + d(p_1 \times p_2^{a_2} \times \dots\dots \times p_{\frac{r+1}{2}}) - 2 + d\left(p_1 \times p_{(\frac{r}{2}+1)}^{a_{(\frac{r}{2}+1)}} \times \dots\dots \times p_r\right) - 2.$

$= (a_2+1)(a_3+1)2^{(r-3)} + (2a_1 + r - 2)2^{(\frac{r}{2}-1)} + (2a_1 + r)2^{(\frac{r}{2}-2)} - 7$.

Remark 3.8. From the preceding theorems it is evident that the cyclic edge connectivity number of an arithmetic graph $G = V_n$ depends not only on the number of primes in *n* but also on the powers of prime. The cyclic edge connectivity number for an arithmetic graph exists for $G = V_n$, $n = p_1^{a_1} \times p_2^{a_2}$, where $a_1 > 2, a_2 \geq 2$. If $a_i \neq 1$ for exactly one *i* then the number of primes in *n* must be greater than or equal to 3 and if $a_i \neq 1$ for at least two *i*, $i \in \{1,2,\dots r\}$ and the number of primes must be greater than or equal to 3. The following theorem gives the cyclic edge connectivity number of an arithmetic graph and we have omitted the above discussed cases and starts only from *r* greater than three and $a_i \neq 1$ for more than three *i.*

Theorem 3.9. For an arithmetic graph $G=V_n$, $n = p_1^{a_1} \times p_2^{a_2} \times \dots\dots \times p_r^{a_r}, r>3 , if\ a_i \neq 1$ then the cyclic edge connectivity number

$$\lambda_c(G) = [\prod_{i=1, i\notin B}^{r} (a_i + 1)] + [|B - B'| + \sum_{i \in B'} a_i] \prod_{i=1, i\notin B}^{r} (a_i + 1) + [|B - B'| + \sum_{i \in B'} a_i] \prod_{i=1, i\notin B}^{r-1} (a_i + 1) - 7 \text{ if r is odd.}$$

$\lambda_c(G) = [\prod_{i=1, i\notin B}^{r} (a_i + 1)] + [|B - B'| + \sum_{i \in B'} a_i] \prod_{i=1, i\notin B}^{r-1} (a_i + 1) + [|B - B'| + \sum_{i \in B'} a_i] \prod_{i=1, i\notin B}^{r} (a_i + 1) - 7$ if r is even. Here *B* is the number of primes in the chosen vertex and B' is the number of $a_i = 1$ in the chosen vertex

Proof.

Case(i) r is odd

Arrange a_i in such a way that $a_1 \geq a_2 \geq a_3 \dots\dots \geq a_r$. Now choose the cycle C_3 such that $V(C_3) = \{ p_1^{a_1}, p_3^{a_3} \times p_5^{a_5} \times \dots\dots \times p_r^{a_r}, p_2^{a_2} \times p_4^{a_4} \times \dots\dots \times p_{r-1}^{a_{r-1}} \}$. Then similar as above theorems, the cycle which is chosen whose degree sum is minimum. Therefore, we have

$$\lambda_c(G) = [\prod_{i=1, i\notin B}^{r} (a_i + 1)] - 1 - 2 + [|B - B'| + \sum_{i \in B'} a_i] \prod_{i=1, i\notin B}^{r} (a_i + 1) - 2 + [|B - B'| + \sum_{i \in B'} a_i] \prod_{i=1, i\notin B}^{r-1} (a_i + 1) - 2$$

$$\lambda_c(G) = [\prod_{i=1, i\notin B}^{r} (a_i + 1)] + [|B - B'| + \sum_{i \in B'} a_i] \prod_{i=1, i\notin B}^{r} (a_i + 1) + [|B - B'| + \sum_{i \in B'} a_i] \prod_{i=1, i\notin B}^{r-1} (a_i + 1) - 7.$$

Case(ii) r is even

Similar as case(i) choose the cycle C_3 whose vertex set $V(C_3) = \{ p_1^{a_1}, p_3^{a_3} \times p_5^{a_5} \times \dots\dots \times p_{r-1}^{a_{r-1}}, p_2^{a_2} \times p_4^{a_4} \times \dots\dots \times p_r^{a_r}\}$. Hence similar as above case we have

$$\lambda_c(G) = [\prod_{i=1, i\notin B}^{r} (a_i+1)] - 1 - 2 + [|B-B'| + \sum_{i\in B'} a_i] \prod_{i=1,i\notin B}^{r-1} (a_i+1) - 2 + [|B-B'| + \sum_{i\in B'} a_i] \prod_{i=1,i\notin B}^{r} (a_i+1) - 2$$

$$\lambda_c(G) = [\prod_{i=1, i\notin B}^{r} (a_i+1)] + [|B-B'| + \sum_{i\in B'} a_i] \prod_{i=1,i\notin B}^{r-1} (a_i+1) + [|B-B'| + \sum_{i\in B'} a_i] \prod_{i=1,i\notin B}^{r} (a_i+1) - 7.$$

IV. Cyclically Optimal Arithmetic Graph

In this section we study the cyclically optimality of arithmetic graphs.[3] For two vertex sets X, Y$\subset V(G)$, [X,Y] is the set of edges with one end in X and other end in Y. Let ω_G(X)=| [X,$\bar{X}$]| is the number of edges between X and $\bar{X}$ in G. If [X,$\bar{X}$] is a minimum cyclic edge cut, then both G[X] and G [$\bar{X}$] are connected. Define ζ=min{ $\omega_G(X)/X$ induces a shortest cycle in G}.

We shall show that any cyclically separable graph G satisfies $\lambda_c(G) \le \zeta(G)$. Hence, a cyclically separable graph G is called cyclically optimal if $\lambda_c(G) = \zeta(G)$. A vertex set X is a cyclic edge fragment, if [X,$\bar{X}$] is a minimum cyclic edge cut. A cyclic edge fragment with minimum cardinality is called a cyclic edge atom. If X is an atom and X' is a proper subset of X such that $[X, \bar{X}]$ is a cyclic edge cut, then $\omega_G(X') > \omega_G(X)$.

Theorem 4.1. Let G=V_n be an arithmetic graph, n = ${p_1}^{a_1} \times {p_2}^{a_2}$; $a_1 \ge a_2 \ge 2$, $a_1 > 2$ then

$$\zeta(G) = \begin{cases} 2a_1 - 1 & if \quad a_1 > 2, a_2 = 2 \\ 2(a_1 + a_2 - 3) & if \quad a_1 \ge a_2 > 2 \end{cases}$$

Proof. Consider an arithmetic graph G=V_n, n= ${p_1}^{a_1} \times {p_2}^{a_2}$

Case(i) $a_1 > 2$ and $a_2 = 2$. Then the vertex set V(G)={ $p_1, {p_1}^2, \dots, {p_1}^{a_1}, p_2, {p_2}^2, p_1 \times p_2, p_1 \times {p_2}^2, \dots . {p_1}^{a_1} \times p_2, {p_1}^{a_1} \times {p_2}^2$ }. By Theorem1.4, G is a bipartite graph shortest cycle is C_4. Let us choose the cycle C_4

$$\zeta(G) = d({p_1}^{a_1}) + d({p_1}^{a_1-1}) + d(p_1 \times p_2) + d(p_1 \times {p_2}^2) - 8.$$

By proof of Theorem 1.3,

$$\zeta(G) = a_2 + a_2 + a_1 + a_2 + a_1 + 1 - 8.$$

$$\zeta(G) = 3a_2 + 2a_1 - 7$$

Since $a_2 = 2$ we get

$$\zeta(G) = 2a_1 - 1.$$

Case(ii) If $a_1 \ge a_2 > 2$, then similar as case (i) choose a cycle C_4 whose degree sum is minimum. Since $a_1 \ge a_2$, then $d({p_1}^{a_1}) \le d({p_2}^{a_2})$. Hence the vertex set of the cycle C_4 whose degree sum minimum is V(C_4)={ ${p_1}^{a_1-1}, {p_1}^{a_1}, p_1 \times {p_2}^{a_2-1}, p_1 \times {p_2}^{a_2}$}. Hence

$\zeta(G) = d({p_1}^{a_1}) + d({p_1}^{a_1-1}) + d(p_1 \times {p_2}^{a_2-1}) + d(p_1 \times {p_2}^{a_2}) - 8.$

by the proof of Theorem1.3, we get $\zeta(G)$ =$a_2 + a_2 + a_1 + 1 + a_1 + 1 - 8$.

=2$(a_1 + a_2 - 3)$.

Theorem 4.2. If G=V_n, is an arithmetic graph, n = ${p_1}^{a_1} \times {p_2}^{a_2}$; where $a_1 = 3$ and $a_2 = 2$ or 3 then G is cyclically optimal.

Proof. Let G=V_n be an arithmetic graph where n = ${p_1}^{a_1} \times {p_2}^{a_2}$

Case(i) If $a_1 = 3, a_2 = 2$ then by Theorem 4.1 and Theorem3.1 we found that $\zeta(G) = 2a_1 - 1 = 5$ and $\lambda_c(G) = 5$. Hence G is cyclically optimal.

Case(ii) If $a_1 = 3, a_2 = 3$ then by Theorem 4.1, we have $\zeta(G) = 2(a_1 + a_2 - 3) = 6$

and by Theorem 3.2, $\lambda_c(G) = 2a_2 + a_1 - 3$ =6. Hence G is cyclically optimal.

Theorem 4.3. In an arithmetic graph G=V_n, $n = {p_1}^{a_1} \times {p_2}^{a_2}; a_1 > 3, a_2 \ge 3$ is cyclically non optimal.

Proof. By Theorem 3.2 and Theorem 4.1, we identified that $\lambda_c(G) \ne \zeta(G)$ for $a_1 > 3$. Hence G is cyclically non optimal.

Remark 4.4

From the theorems in section 3, we found that in an arithmetic graph G=V_n, $n = {p_1}^{a_1} \times {p_2}^{a_2} \times \dots . \times {p_r}^{a_r}, r \ge 3$, $a_i \ge 1$ for $i \in \{1,2,3, \dots, r\}$ the cyclic edge connectivity number $\lambda_c(G)$ is equal to $\zeta(G)$. Hence these graphs are cyclically optimal.

Conclusion

As per the above theorems, we found that for an arithmetic graph G=V_n the number of primes in n does not exceed 2 is of infinite cyclic edge connectivity. Also, for an arithmetic graph G=V_n, $n = {p_1}^{a_1} \times {p_2}^{a_2}$, where $a_1 \ge a_2 > 2$ the cyclic edge connectivity number $\lambda_c(G) = 2a_2 + a_1 - 3$.

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