Estimation of Age Replacement Policy when Maintenance Cost is Linear and Nonlinear

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Abstract: The classical solution for replacement problem requires complete knowledge about the component life distribution. In this paper, suitable estimator of the age replacement strategy has been suggested and it is shown that the estimator is strongly consistent. A simulation study has been carried out to illustrate the efficiency of the estimator.

Index Terms: Age replacement policy, increase failure rate (IFR), kernel estimator, maintenance cost, strong consistency, rayleigh distribution.

I. INTRODUCTION

Consider a system which is subject to failure due to failure of the components. The components can be replaced suitably and instantaneously in order to keep the system functional. Let the life distribution function of the component be \( F: \mathbb{R}^+ \to [0, 1] \), which is absolutely continuous having density function \( f \) and survival function \( \bar{F} \equiv 1 - F \). Generally, a replacement policy involves routine replacement (on failure) and preventive replacement (in anticipation of failure). The mostly discussed replacement policies are the age replacement policy and the block replacement policy (cf. Arunkumar (1972), Ascher and Feingold (1984), Rigdon and Basu (2000)). In age replacement policy (see Bergman (1979), Frees and Ruppert (1985), Ingram and Scheaffer (1976), Jiang and Ji (2002), Lim, Qu and Zuo (2016), Park, Jung and Park (2015) and Yeh, Chen and Chen (2005)), a component is replaced by a new component of the same type on failure or after a specified age \( T \), whichever is earlier. In periodic replacement policy, replacement is done by a new equipment of the same type at specified equidistant points of time \( T, 2T, \ldots \) and only minimal repair is undertaken for any intervening failure in order to keep the system functional. By minimal repair we mean the system will be repaired in such a way that the failure rate of the system will remain same as that just prior to its failure. Roy and Basu (1993) have considered the estimation of age and periodic replacement policies, where they have taken only the replacement cost against failure and the planned replacement cost. Here we have considered the estimation of the age replacement policy taking maintenance cost of the system at different points of time into consideration, since it plays an important role in replacing the system. In most of the cases, it happens that after certain age of the system, though the system is functioning, its maintenance cost goes role in replacing the system. In most of the cases, it happens that after certain period of time, where they have taken only the replacement cost against failure and its maintenance cost goes.

Let \( \tau_i \) be the age at which the \( i \)-th component fails, \( i = 1, 2, \ldots \). Then the failure rate (or hazard rate) function \( \lambda(t) \) is defined as

\[
\lambda(t) = \frac{f(t)}{\bar{F}(t)} = \lim_{t \to 0} \frac{P(t < \tau < t + \Delta t)}{\Delta t},
\]

where \( \bar{F}(t) = 1 - F(t) \) is the survival function of the system at time \( t \).

Let \( T \) be the optimal age replacement time such that \( T \) is a function of the components' lifetimes and their maintenance costs. The optimal age replacement time is given by

\[
T = \min_{t \geq 0} \{ t : \int_0^t \lambda(u) du = \int_0^t f(u) du \},
\]

where \( \lambda(t) = \frac{f(t)}{\bar{F}(t)} \) is the hazard rate function and \( f(t) \) is the density function of the system's lifetime.

II. ESTIMATION AND THE PROPERTIES OF THE ESTIMATOR

A. Kernel Estimation

Let \( X_1, X_2, \ldots \) be a sequence of independent and identically distributed random variables having distribution function \( F(x) \), survival function \( S(x) = 1 - F(x) \), and density function \( f(x) = \frac{dF(x)}{dx} \). Consider a kernel function \( K(x) \) such that

\[
(i) \quad K(x) \text{ is uniformly continuous on } R = (-\infty, \infty).
\]

\[
(ii) \quad K(x) \text{ is} \text{of bounded variation on } R.
\]

\[
(iii) \quad \text{There exists a } \zeta \text{ such that } K(x) = 0 \text{ for } |x| > \zeta.
\]

\[
(iv) \quad \int_{-\infty}^{\zeta} K(x) dx = 1.
\]

\[
(v) \quad \int_{-\infty}^{\zeta} |K(x)| dx < \infty.
\]

\[
(vi) \quad \int_{-\infty}^{\zeta} |x| |K(x)| dx < \infty.
\]

Define \( f_n(x) \) as

\[
f_n(x) = \frac{1}{nh_n} \sum_{i=1}^{n} K \left[ \frac{x - X_i}{b_n} \right],
\]

where \( K \) is a kernel function, \( n \) is the number of observations, \( h_n \) is the bandwidth, and \( b_n \) is the smoothing parameter.
where \( b_n(>0) \), the window width (or bandwidth) is taken as the \( n^{th} \) term of the sequence of positive numbers assumed to satisfy the following conditions (cf. Silverman (1978)).

(a) \( b_n \to 0 \) as \( n \to \infty \).

(b) \( \frac{b_n}{\ln n} \to \infty \) as \( n \to \infty \).

For further discussion on kernel density estimation, one can refer to Akaishi (1954), Prakasa Rao (1983), Rosenblatt (1956), Schuster (1969), Van Ryzin (1969) to name a few.

Let us take \( g(i,T) = \alpha + \beta i \), for \( i = 1, 2, \ldots \), with \( T \geq 0 \) and \( \{N(t), t \geq 0\} \), be a point process with \( E[N(t)] = \delta t \). It is to be noted that we do not assume any specific stochastic process for \( N(t) \). Here the maintenance cost is linear in \( t \). For \( T \in [t_k, t_{k+1}], g(i,T) = \alpha + \beta i \). This means, in each such interval, \( g(i,T) \) is constant. That is, it is linear in \( i \), but constant in \( T \). In other words, the maintenance cost is a piecewise constant function. Then (1) reduces to

\[
L(T) = \int_0^T \tilde{F}(u)du \left[ f(T) + (\alpha + \beta \lambda T)\tilde{F}(T) \right] - \frac{\lambda + F(T)}{R_1} \left( \alpha \int_0^T \tilde{F}(u)du + \beta \int_0^T u\tilde{F}(u)du \right) \tilde{F}(T) = 0.
\]

Note that \( T = \infty \) is always a solution to (2).

Define

\[ L^*(T) = \frac{L(T)}{L(T)} \]

If \( F \) is IFR, \( L^*(T) \) is increasing in \( T \) and must have atmost one finite root. Write

\[
F_n(x) = 1 - F_n(x) = \int_0^x f_n(t)dt
\]

and

\[
L_n(T) = \int_0^T \tilde{F}(u)du \left[ f_n(T) + (\alpha + \beta \lambda T)\tilde{F}(T) \right] - \frac{\lambda + F_n(T)}{R_1} \left( \alpha \int_0^T \tilde{F}(u)du + \beta \int_0^T u\tilde{F}(u)du \right) \tilde{F}(T) = 0.
\]

Let

\[
T_n = \inf \{ T : L_n(T) = 0 \} = \min \{ T : L_n(T) = 0 \}.
\]

**Theorem 1:** If \( F \) is IFR having uniformly continuous density function \( f(\cdot) \), then \( T_n \), defined in (4), is a strongly consistent estimator of the optimal age replacement policy \( T_0 \).

**Proof:** Define

\[
\delta_n = \sup \{ |F_n(x) - f(x)| : x \geq 0 \}.
\]

Observe that

\[
|F_n(x) - f(x)| \leq x\delta_n.
\]

**Case 1:** Let \( T_n < \infty \). First we show that the sequence \( \{T_n\} \), defined in (4) is bounded above. For this, if possible, let \( \{T_n\} \) be unbounded. So, for every positive number \( M \),

\[
T_n > M \text{ infinitely often (i.o.)}.
\]

This further gives

\[
L_n(M) \neq 0 \quad \text{i.o.}
\]

But \( L_n(0) = -\lambda(0) \). So, \( L_n(M) < 0 \) for every \( M > 0 \), since \( L_n(\cdot) \) is continuous. Thus we have, for every \( M > 0 \), there exists an increasing sequence of indices \( \{n_k\} \) such that for every \( k \)

\[
\int_0^M \tilde{F}(u)du \left[ f_{n_k}(M) + (\alpha + \beta \lambda M)\tilde{F}_{n_k}(M) \right] - \lambda + F_{n_k}(M) + \frac{\lambda}{R_1} \left( \alpha \int_0^M \tilde{F}(u)du + \beta \int_0^M u\tilde{F}(u)du \right) \tilde{F}_{n_k}(M) < 0.
\]

Taking limit as \( k \to \infty \), we have

\[
\int_0^M \tilde{F}(u)du \left[ f(M) + (\alpha + \beta \lambda M)\tilde{F}(M) \right] - \lambda + F(M) + \frac{\lambda}{R_1} \left( \alpha \int_0^M \tilde{F}(u)du + \beta \int_0^M u\tilde{F}(u)du \right) \tilde{F}(M) < 0.
\]

Or, for every \( M(>0) \),

\[
\phi(M) = \int_0^M \tilde{F}(y)dy - \frac{\lambda + F(M)}{R_1} \left( \alpha \int_0^M F(u)du + \beta \int_0^M uF(u)du \right) \tilde{F}(M) < 0.
\]

where \( x(\cdot) \) is the failure rate corresponding to \( F(x) \) at the point \( x \). Taking derivative of the left-hand side of (7) with respect to \( M \), we have

\[
\phi'(M) = -\frac{\lambda + F(M)}{R_1} \int_0^M \tilde{F}(y)dy \geq 0,
\]

since \( F \) is IFR. If \( T_0 \) be a solution of (2), then \( \phi(M) > 0 \) for all \( M > T_0 \), which contradicts (7). Thus, there exists an \( M(>0) \) such that \( T_n \leq M \) for all \( n \).

Now,

\[
|L(T_n)| = |L(T_n) - L_n(T_n)| \leq t_{1n} + t_{2n} + \ldots + t_{5n},
\]

where

\[
t_{1n} = \left| f(T_n) \int_0^T F(u)du - f_n(T_n) \int_0^T F_n(u)du \right|,
\]

\[
t_{2n} = \left| F(T_n)\tilde{F}(T_n) - F_n(T_n)\tilde{F}(T_n) \right|,
\]

\[
t_{3n} = \lambda \left| F(T_n) - F_n(T_n) \right|,
\]

\[
t_{4n} = \frac{\beta \delta T n}{R_1} \left| \tilde{F}(T_n) - \tilde{F}_n(T_n) \right| \int_0^T F(u)du - \tilde{F}(T_n) \int_0^T F_n(u)du,
\]

and

\[
t_{5n} = \frac{\beta \delta T n}{R_1} \left| \tilde{F}(T_n) - \tilde{F}_n(T_n) \right| \int_0^T u\tilde{F}(u)du - \tilde{F}(T_n) \int_0^T u\tilde{F}_n(u)du.
\]

Noting that \( L(T_n) = \int_0^T u\tilde{F}(u)du - \int_0^T u\tilde{F}_n(u)du \)

\[
\int_0^T F(u)du - \int_0^T F_n(u)du + \int_0^T F(u)du - \int_0^T F_n(u)du,
\]

\[
\leq \delta_n \int_0^T (F(u)du - \tilde{F}(u)du) + \delta_n \int_0^T (F_n(u)du - \tilde{F}_n(u)du)
\]

\[
\leq \left( \mu + M \right) \int_0^T (\tilde{F}(u)du - \tilde{F}_n(u)du)
\]

\[
\leq \left( \mu + M \right) \left( 1 + \frac{\lambda M^2 \beta \delta T }{R_1} \right) \delta_n.
\]

The first inequality follows from (5) and (6), second inequality follows due to the fact that \( \int_0^T F(u)du \leq \mu = \int_0^T F_n(u)du \) and \( T_n \leq M \) for all \( n \), whereas the last inequality can be obtained as under-

Since \( T_n \) is a solution of (3), we have

\[
f_n(T_n) \int_0^T F_n(u)du = F_n(T_n) \left[ \lambda + F_n(T_n) \right] + \frac{\beta \delta T n}{R_1} \left| \int_0^T u\tilde{F}(u)du - \int_0^T u\tilde{F}_n(u)du \right|
\]

which is equivalent to

\[
f_n(T_n)T_n \leq \left( \mu + M \right) \left( 1 + \frac{\lambda M^2 \beta \delta T }{R_1} \right) \delta_n + \left( \mu + M \right) \left( 1 + \frac{\lambda M^2 \beta \delta T }{R_1} \right) \delta_n.
\]
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III. ESTIMATION AND THE PROPERTIES OF THE ESTIMATOR WHEN MAINTENANCE COST IS NOT PIECEWISE CONSTANT

In practice maintenance cost need not increase linearly. To cope up this situation, in this section we analyze the case when the maintenance cost is a quadratic function. The general case when the maintenance cost is a higher order polynomial can be analyzed similarly.

Let us take \( g(i, T) = \alpha + i\beta + i^2\gamma \), for \( i = 1, 2, \ldots \), with \( T \geq 0 \) and \( \{N(t), t \geq 0\} \), a point process with \( E[N(0)] = \delta t \). Here \( \alpha, \beta, \gamma \) are nonnegative because of the following reason:

Let \( f(x) = a + bx + cx^2 \) for all \( x > 0 \) and \( f(x) \) be nondecreasing for all \( x \). Then \( f'(x) = 2cx + b \geq 0 \) \( \Rightarrow b \geq 0 \) (otherwise, as \( x \to 0, f'(x) < 0 \)). Further, \( f(x) > 0 \) for all \( x > 0 \) \( \Rightarrow b^2 - 4ac > 0 \) and \( c \) will be of same sign. Again, \( f(0) \geq 0 \) \( \Rightarrow a \geq 0 \). Hence \( a, b, c \geq 0 \).

Now (1) reduces to

\[
T_{\ast} = \frac{-f(T) + \sqrt{f(T)^2 - 4f(0)f(T_0)}}{2f(0)},
\]

Thus, \( T_{\ast} \) is continuous. This contradicts our hypothesis that \( T_0 = 0 \). Hence the result.

IV. A SIMULATION STUDY

We conclude our discussion with a simulation study to illustrate the efficiency of our estimation procedure compare to that proposed by Roy and Basu (1993) in the context of an age replacement policy. Since Rayleigh distribution has a wide range of applications in modeling life distributions of various electronic components, we take this distribution having survival function

\[ F(x) = e^{-\eta x^2}; \quad \eta > 0, x \geq 0, \]

as the underlying distribution. This is a strictly increasing failure rate (IFR) life distribution. For specific set of values of \( \eta, \lambda, \alpha, \beta, \gamma \) and \( R_1 \), we can calculate the optimal age replacement time \( T_0 \) by solving equation (2). On the basis of the random observations generated from Rayleigh distribution, we calculate the Roy and Basu's estimator \( RB_{\ast} \), as well as the estimator \( T_{\ast} \) as defined in (4). Then we measure the absolute difference of \( T_0 \) from \( RB_{\ast} \) and also from \( T_{\ast} \) respectively. To increase the level of efficiency, the entire process is repeated 100 times. We note the proportion of occasions for which \( |T_0 - T_{\ast}| < |T_0 - RB_{\ast}| \) and consider this proportion as a measure of efficiency of \( T_{\ast} \) over \( RB_{\ast} \) in estimating \( T_0 \).

Let

\[ T_{\ast} \quad \text{def} = \inf \{ T : L^*_{\ast}(T) = 0 \} \]

The following theorem shows that the sequence \( T_{\ast} \) is strongly consistent under certain condition. The proof is similar to that of Theorem 1 with obvious modifications and is not given here.

Theorem 2: If \( F \) is IFR having a uniformly continuous density function \( f \), then \( T_{\ast} \), defined in (9), is a strongly consistent estimator of the optimal age replacement policy \( T_0 \).

Remark 1: One can show that the sequence \( T_{\ast} \) is strongly consistent under the condition above the theorem when the maintenance cost is a piecewise polynomial of degree \( m \).
and the window width \( b_n = \sqrt{n} \). It is to be noted that the kernel estimator \( f_n \) of the underlying density function is piecewise continuous and vanishes outside \([X_{(min)} - \gamma b_n, X_{(max)} + \gamma b_n]\). We adopt an iterative method to solve the integral equations (2) and (3) and stop the iteration as soon as the difference of two consecutive solutions falls below \( 10^{-6} \).

On the basis of 100 runs each involving 40 random observations, generated from the underlying distribution, the efficiency of the proposed estimator \( T_n \) is illustrated in Table 1. This shows that the estimator \( T_{40} \) \((T_n\) based on \( n = 40 \)) proposed in this paper is better than Roy and Basu’s estimator \( RB_{40} \) for all the values of the parameters considered here. Column 7 of this table (E) indicate Efficiency of \( T_n \) over \( RB_n \).

### Table 1

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### REFERENCES


