Approximate Optimum Strata Boundaries for Equal Allocation under Ranked Set Sampling

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Abstract: Optimum stratification is the method of choosing the best boundaries that make strata internally homogeneous, given some sample allocation. In order to make the strata internally homogenous, the strata should be constructed in such a way that the strata variances for the characteristic under study be as small as possible. This could be achieved effectively by having the distribution of the study variable known and create strata by cutting the range of the distribution at suitable points. If the frequency distribution of the study variable is unknown, it may be approximated from the past experience or some prior knowledge (auxiliary information) obtained at a recent study. In this study, problem of optimum stratification on the auxiliary variable \( x \) for equal allocation has been considered. A \( \text{cum}\sqrt{K_3(x)} \) rule of obtaining approximately optimum strata boundaries has been proposed under ranked set sampling. A numerical investigation with relative efficiency has also been made.

Index Terms: Ranked set Sampling, approximately optimum strata boundaries, auxiliary variable, optimum strata width

INTRODUCTION

In sample survey, the precision of an estimator of a population parameter depends on the heterogeneity of the units of the population besides the sample size and sampling fraction, the role of stratified sampling method comes into play as one possible way to enhance the precision of the estimator. In stratified sampling, a heterogeneous population is divided into a number of strata so as to increase the homogeneity among population units within strata and then a sample is drawn from each stratum by using any suitable sample selection method. The main aspects which are to be dealt with tactically for enhancing precision of an estimator of a population parameter are construction of strata, number of strata to be made, allocation of sample size to strata and stratification variable(s). In the construction of strata, the major concerns are determination of optimum strata boundaries and choice of the best characteristic.

Samawi (1996) introduced concept of stratified ranked set sample (SRSS) and define it a sampling plan in which a population is divided into L mutually exclusive and exhaustive strata and a ranked set sample (RSS) of \( n \) elements is quantified within each stratum \( h \). The sampling is performed independently across the strata. Therefore, we can think of a SRSS scheme as a collection of \( L \) separate ranked set samples. McIntyre (1952) introduced the concept of RSS.

Let the population under consideration be divided into \( L \) strata and a sample \( n_{0h} = (R_h \times n_h) \) units is selected from \( h^{th} \) stratum is drawn using RSS, where \( R_h \) is the number of cycles and \( n_h \) is sample size of each cycle. Each sample element is measured with respect to some variable \( Y \), and estimator of the population mean is given by

\[
\bar{y}_{SRSS} = \sum_{h=1}^{L} \frac{W_h}{n_{0h}} \left[ \sum_{j=1}^{R_h} \sum_{i=1}^{n_h} \bar{y}_{ij(i)} \right]
\]

(1.1)

where \( W_h \) is the weight of units in the \( h^{th} \) stratum and \( \bar{y}_{ij(i)} \) is the sample mean based on \( n_{0h} \) units drawn from the \( h^{th} \) stratum.

If finite population correction is ignored, then the minimization of the variance expression

\[
\frac{1}{n} \sum_{h=1}^{L} W_h^2 \sigma_{h(i)}^2 + \mu_{h(i)}^2
\]

is equivalent to the minimization of

\[
\sum_{h=1}^{L} W_h^2 \sigma_{h(i)}^2
\]

(1.2)

\[
\sigma_{h(i)}^2 = \left( \sigma_{hc}^2 - \frac{1}{n} (\mu_i - \mu)^2 \right)
\]

where \( \sigma_{h(i)}^2 \) represents the variance of \( i^{th} \) order statistics in \( h^{th} \) stratum of the random sample size \( n_h \).

For given \( L \), \( n \) and the variable \( y \), the variance in (1.2) are functions of the strata boundaries. The boundaries that correspond to the minimum

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of variance for any particular allocation method are called the optimum strata boundaries (OSB). The problem of optimum stratification on the study variable for both proportional and Neyman allocation was first considered by Hayashi and Murayama (1951) and sub sequently by Dalenius (1950). Sometimes the information about the study variables are not known, the utilization of auxiliary as stratification variable was considered by Singh and Sukhatme (1969). Further problem of optimum stratification for equal allocation under simple random sampling has been done by Singh and Parkash (1973). Rizvi, et al. (2000) developed the theoretical framework for determination of optimum strata boundaries on an auxiliary variable (X) closely related with the study variables, appropriate for surveys involving two study variables. Khan (2008) problem of finding OSB is considered as the problem of determining Optimum Strata Widths (OSW). Danish and Rizvi (2018) proposed a technique under Neyman allocation when the stratification is done by utilizing two auxiliary variables as stratification variables. Rizvi et al. (2002), Verma and Mahajan (2004), Mehta and Mandowara (2012), Rao et al. (2014), Fonolahi and Khan (2014), Hidiroglou and Kozak (2017), Danish et al. (2017), Danish, F. (2018), Khan and Reddy (2019), also consider different allocation problems. A recent study on the problem of optimum stratification for a model-based allocation in a super population model has been made by Gupta and Ahamed (2021). Rather et al. (2022) studied the problem of optimum stratification using RSS.

One of the various methods of finding AOSB on the study variable \( y \) is due to Dalenius and Gurney [2]. According to this method the strata boundaries for which \( W_h \sigma_{hy} \) becomes same for all strata are the AOSB for the Neyman allocation. Motivated by these studies, in this paper we deal with the problem of determining AOSB for equal allocation using ranked set sampling that has different frequency distributions.

1. MINIMAL EQUATIONS

If the regression of the estimation variable \( 'y' \) on the stratification variable \( 'x' \), in the infinite super population is given by

\[
y = c(x) + e
\]

Where \( c(x) \) is a function of auxiliary variable, \( e \) is the error term such that \( E(e \mid x) = 0 \) and \( V(e \mid x) = \eta(x) > 0 \) \( \forall x \in (a, b) \) with \( b - a < \infty \). Let \( f(x, y) \) be the joint density function of \((x, y)\) and \( f(x) \) be the marginal density function of \( x \).

Then, we have the following relations:

\[
W_h = \int_{x_{h-1}}^{x_h} f_i(x) \, dx \), \quad \mu_{hc} = \frac{1}{W_h} \int_{x_{h-1}}^{x_h} c(x) f_i(x) \, dx \quad \text{and} \quad \sigma_{hy}^2 = \sigma_{hc}^2 + \mu_{hq} \quad (h = 1, 2, 3, ..., L)
\]

where \((x_{h-1}, x_h)\) are lower and upper boundaries of the \( h^{th} \) stratum with \( x_0 = a \) and \( x_L = b \). \( \mu_{hq} \) is the expected value of \( \eta(x) \) and \( \sigma_{hc}^2 \) is the variance of \( c(x) \) in the \( h^{th} \) stratum.

Using these relations, the variance expression are reduced to

\[
V(y_{SRSS})_{eq} = \left[ \sum_{h=1}^{L} W_h^2 \sigma_{h(i)}^2 \right]
\]

(2.3)

Let \([x_h]\) denote the set of optimum points of stratification on the range \((a, b)\), for which the \( V(y_{SRSS}) \) is minimum. These points \([x_h]\) are the solutions of the minimal equations which are obtained by equating to zero the partial derivatives of \( V(y_{SRSS}) \) with respect to \([x_h]\). We shall now obtain these minimal equations for proportional allocations.

To obtain the minimal equations for this allocation method, we minimize the variance expression given in (2.3) w.r.t. \( x_h \), and get minimal equations as

\[
w_h \left\{ \left( c(x_h) - \mu_{hc} \right)^2 - \sigma_{hc(i)}^2 + \eta(x_h) - \mu_{hq} \right\}
\]

\[
= w_i \left\{ \left( c(x_i) - \mu_{ic} \right)^2 - \sigma_{ic(i)}^2 + \eta(x_i) - \mu_{iq} \right\}
\]

(2.5)

where \( i = h + 1, h = 1, 2, ..., L - 1 \)

These equations are also complicated to solve and therefore could be used to find stratification points. Further, better approximation can be obtained by using some approximate iterative procedures.

2. APPROXIMATE SOLUTIONS OF THE MINIMAL EQUATIONS

In this section, we shall find the series expansions of the system of equations given in (2.5) about the point \([x_h]\), the common boundary of \( h^{th} \) and \((h + 1)^{th} \) strata and obtain their approximated solutions. To find the expressions of the left hand side of (2.5) we shall use the relations obtained in different lemmas by replacing \((y, x)\) by \( (x_{h-1}, x_h) \), and for the right side corresponding relations after replacing \((y, x)\) by \((x_{h-1}, x_h)\) will be used.

Let us first consider the development of right hand side. The corresponding expansion for the left hand side of the equation can be obtained from the expansion of right side by merely changing the signs of the coefficients of odd powers of \( k \), where \( k = (x_{h+1}, x_h) \), although the same result will be obtained, we have

\[
\mu_h(y, x) = \eta \left[ 1 + \frac{\eta f}{2 \eta} + \frac{\eta f + 2 f f'}{12 \eta} \right] k^2
\]

\[
+ \frac{\eta f f' + \eta f' f'' + f^2 f' f'''}{24 f^2 \eta} k^3 + O(k^4)
\]

(2.6)

\( O(k^i) \) is the higher order terms with power \( \geq i \)

from above Eq. after replacing \( 'y' \) and \( 'x' \) by \( x_{h} \) and \( x_{h+1} \) respectively.
\[
\mu_{ic} = c \left[ 1 + \frac{c^2}{2c} k_i + \frac{c f^2 + 2 f c''}{12 f c} k_i^2 + \frac{f f' c + f f' c'' + f^2 c'''}{24 f^2 c} k_i^3 + O(k_i^4) \right]
\]

where the function \( c \), \( f \) and their derivatives are evaluated at \( x_h \).

Therefore, we get

\[
\left[ \mu_{ic} - c(x_h) \right]^2 = \left( \frac{k_i^2}{4} \right) \left[ c^2 + \frac{c^2}{3} + \frac{c f c''}{k_i} + O(k_i^2) \right]
\]

also, we have

\[
\sigma_{ic}^2 = \frac{k_i^2}{4} \left[ c^2 + \frac{c^2}{3} + \frac{c f c''}{k_i} + O(k_i^2) \right]
\]

\[
[\mu_{ic} - c(x_h)]^2 + \sigma_{ic}^2 = \frac{k_i^2}{12} \left[ 4c^2 + \frac{c^2}{f} + \frac{3f c c''}{k_i} \right] + O(k_i^3)
\]

\[
\eta(x_h) + \mu_{i\eta} = \eta \left[ 2 + \frac{n f}{2\eta} k_i + \frac{(n f + 2 n f'')}{12 f n} k_i^2 + \frac{(f f' n + f f' n' + f^2 n'' - n f')}{24 f^2 n} k_i^3 + O(k_i^4) \right]
\]

Where on the right side of the above equality, the functions \( \eta \), \( f \) and their derivatives are evaluated at the point \( x_h \).

Thus, we get

\[
W_i \left[ \mu_{ic} - c(x_h) \right]^2 + \sigma_{ic}^2 + \eta(x_h) + \mu_{i\eta} = 2\sqrt{n} \left[ f \sqrt{\eta k_i} + \frac{(n f + 2 f n')}{4\sqrt{n}} k_i^2 + O(k_i^3) \right]
\]

\[
\Rightarrow W_i \left[ \mu_{ic} - c(x_h) \right]^2 + \sigma_{ic}^2 + \eta(x_h) + \mu_{i\eta} = \int_{x_h}^y \sqrt{g(t) f_i(t) dt} \left[ 1 + O(k_i^2) \right]
\]

Similarly from (2.5), we have

\[
W_h \left[ \mu_{hc(i)} - c(x_h) \right]^2 + \sigma_{hc(i)}^2 + \eta(x_h) + \mu_{i\eta} = 2\sqrt{n} \left[ f \sqrt{\eta k_i} + \frac{(n f + 2 f n')}{4\sqrt{n}} k_i^2 + O(k_i^3) \right]
\]

\[
\Rightarrow W_h \left[ \mu_{hc(i)} - c(x_h) \right]^2 + \sigma_{hc(i)}^2 + \eta(x_h) + \mu_{i\eta} = \int_{x_h}^y \sqrt{g(t) f_i(t) dt} \left[ 1 + O(k_i^2) \right]
\]

where \( i = h + 1, h = 1, 2, ..., L - 1 \)

**THEOREM:-**

If the regression of the estimation variable \( 'Y' \) on the stratification variable \( 'X' \), in the infinite super population is given by

\[
y = c(x) + e
\]

where \( 'c(x)' \) is a function of auxiliary variable, \( 'e' \) is the error term such that \( E(e \mid x) = 0 \) and \( V(e \mid x) = \eta(x) \) \( \forall x \in (a, b) \) with \( (b - a) < \infty \), and further if the function \( g_i(x) f_i(x) \in \Omega \), then the system of equations (2.4) given strata boundaries \( (x_h) \) which correspond to the minimum of \( V(\bar{f}_{x_h}) \) can be written as

\[
\int_{x_h}^y \sqrt{g(t) f_i(t) dt} \left[ 1 + O(k_i^2) \right] = \int_{x_h}^y \sqrt{g(t) f_i(t) dt} \left[ 1 + O(k_i^2) \right]
\]

Therefore, if we have sufficiently large number of strata, the \( k_i \)'s (strata widths) are small and their higher powers in the expansion can be neglected, then the system of equations (3.4) can be approximated by

\[
\int_{x_h}^y \sqrt{g(t) f_i(t) dt} = \text{constant}, \ h = 1, 2, ..., L
\]

Where terms of order \( O(Sup_{(a,b)}(k_h)) \) have been neglected on both sides of the equation (3.3)
and \( Q_3(x_{h-1}, x_h) \) is of order \( O(\text{Sup}_{(a,b)}(k_h)) \), then the minimal equations (2.5), can to the same degree of approximation as involved in (3.5), be put as
\[
Q_3(x_{h-1}, x_h) = \text{constant} \cdot h = 1, 2, ..., L
\]
(3.6)
Or equivalently by
\[
Q_3(x_{h-1}, x_h)\left[1 + O(k_i^2)\right] = \int_{x_{h-1}}^{x_i} \sqrt{g(t)} f_i(t) dt
\]
i = h + 1, h = 1, 2, ..., L
Since \( \int_{x_{h-1}}^{x_i} \sqrt{g(t)} f_i(t) dt = O(m) \) when the function \( \sqrt{g(t)} f_i(t) \) is bounded \( \forall x \in (a, b) \). Thus we get the following \( \text{cum}\sqrt{k_3(x)} \) rule for finding AOSB for equal allocation method.

3. \( \text{cum}\sqrt{k_3(x)} \) Rule

If the function \( \sqrt{K_3(x)} = g(t)f_i(t) \) is bounded and its first two derivative exists for all \( x \) in \([a,b]\), then for given value of \( L \) taking equal intervals on the \( \text{cum}\sqrt{K_3(x)} \) rule will give us AOSB.

4. EMPIRICAL STUDY

For the purpose of illustrating the usefulness of the approximate solutions to the minimal equations giving optimum points of stratification, we shall consider following density functions for \( X \). For clarity, the linear regression line of \( 'Y' \) on 'X' of the form \( y = \alpha + \beta x + e \), assuming the value of \( \beta = 0.5 \). For the conditional variance function, it is assumed to have two different forms like the first form could be a constant and the second could be function of auxiliary variable. i.e. \( \eta(x) = \delta \) and \( \eta(x) = \lambda x \), where \( \delta \) and \( \lambda \) are constants.

For the empirical studies under equal allocation let us assume small values of \( \delta = 0.0214 \), \( \lambda = 0.00437 \), such that there may be very small effect of these constants over the estimation.

<table>
<thead>
<tr>
<th>L</th>
<th>AOSB</th>
<th>Total Variance ( V(Y_{SRSS}) )</th>
<th>R.E. %</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.4854</td>
<td>0.75522</td>
<td>102.067</td>
</tr>
<tr>
<td>3</td>
<td>1.3385, 1.6479</td>
<td>0.75233</td>
<td>102.459</td>
</tr>
<tr>
<td>4</td>
<td>1.2565, 1.4934, 1.7593</td>
<td>0.75132</td>
<td>102.597</td>
</tr>
<tr>
<td>5</td>
<td>1.1951, 1.3866, 1.5803, 1.7931</td>
<td>0.75083</td>
<td>102.664</td>
</tr>
<tr>
<td>6</td>
<td>1.1632, 1.3333, 1.5056, 1.7183, 1.8283</td>
<td>0.75063</td>
<td>102.691</td>
</tr>
</tbody>
</table>

If the stratification variable follows the uniform distribution with pdf \( f(x) = \frac{1}{b-a} \), \( x \in [1,2] \), utilizing the \( \text{cum}\sqrt{k_3(x)} \) rule, we get the stratification points as given in Table I.

### Table I: AOSB and Variance when the auxiliary variable is uniformly distributed

\[
\eta(x) = \delta
\]

<table>
<thead>
<tr>
<th>L</th>
<th>AOSB</th>
<th>Total Variance ( V(Y_{SRSS}) )</th>
<th>R.E. %</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.3884</td>
<td>0.67054</td>
<td>101.494</td>
</tr>
<tr>
<td>3</td>
<td>1.2607, 1.4359</td>
<td>0.66863</td>
<td>101.785</td>
</tr>
<tr>
<td>4</td>
<td>1.1975, 1.4041, 1.6548</td>
<td>0.66771</td>
<td>101.925</td>
</tr>
<tr>
<td>5</td>
<td>1.1487, 1.2285, 1.4921, 2.1616</td>
<td>0.66756</td>
<td>101.948</td>
</tr>
<tr>
<td>6</td>
<td>1.1215, 1.2489, 1.3937, 1.5546, 1.7333</td>
<td>0.66714</td>
<td>102.012</td>
</tr>
</tbody>
</table>

If the stratification variable follows the exponential distribution with pdf \( f(x) = e^{-x+1} \), \( x \in [1,5] \), utilizing the \( \text{cum}\sqrt{k_3(x)} \) rule, we get the stratification points as given in Table III.

### Table III: AOSB and Variance when the auxiliary variable is exponentially distributed

\[
\eta(x) = \delta
\]

<table>
<thead>
<tr>
<th>L</th>
<th>AOSB</th>
<th>Total Variance ( V(Y_{SRSS}) )</th>
<th>R.E. %</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2.3821</td>
<td>0.99639</td>
<td>113.661</td>
</tr>
<tr>
<td>3</td>
<td>1.8741, 2.8433</td>
<td>0.96949</td>
<td>116.815</td>
</tr>
<tr>
<td>4</td>
<td>1.7032, 2.4632, 3.0949</td>
<td>0.96076</td>
<td>117.876</td>
</tr>
<tr>
<td>5</td>
<td>1.4911, 2.0471, 2.7491, 3.7337</td>
<td>0.95395</td>
<td>118.718</td>
</tr>
<tr>
<td>6</td>
<td>1.3917, 1.8381, 2.3633, 2.9074, 3.7467</td>
<td>0.95131</td>
<td>81.0282</td>
</tr>
</tbody>
</table>

If the stratification variable follows the exponential distribution with pdf \( f(x) = \frac{2(2-x)}{(b-a)^2} \), using \( a = 1 \) and \( b = 2 \), utilizing the \( \text{cum}\sqrt{k_3(x)} \) rule, we get the stratification points as given in Table II.

### Table II: AOSB and Variance when the auxiliary variable is Right-triangular distributed

\[
\eta(x) = \delta
\]
If the stratification variable follows the standard normal distribution with pdf  
\[ f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad x \in [0,1] \], utilizing the  
cum\sqrt{k(x)} rule, we get the stratification points as given in Table IV.

**Table IV: AOSB and Variance when the auxiliary variable is Standard normally distributed**

<table>
<thead>
<tr>
<th>L</th>
<th>AOSB</th>
<th>Total Variance [\sqrt{V(y_{SRSS})_{eq}}]</th>
<th>R.E. %</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.4901</td>
<td>0.08576</td>
<td>113.307</td>
</tr>
<tr>
<td>3</td>
<td>0.3086, 0.6313</td>
<td>0.07920</td>
<td>122.688</td>
</tr>
<tr>
<td>4</td>
<td>0.2296, 0.4689, 0.7326</td>
<td>0.07893</td>
<td>123.114</td>
</tr>
<tr>
<td>5</td>
<td>0.1857, 0.3706, 0.5768, 0.7812</td>
<td>0.07877</td>
<td>123.360</td>
</tr>
<tr>
<td>6</td>
<td>0.1560, 0.3058, 0.4671, 0.6381, 0.8083</td>
<td>0.07868</td>
<td>123.496</td>
</tr>
</tbody>
</table>

If the stratification variable follows the uniform distribution with pdf  
\[ f(x) = \frac{1}{b-a}, \quad x \in [1,2] \], utilizing the cum\sqrt{k(x)} rule, we get the stratification points as given in Table V.

**Table V: AOSB and Variance when the auxiliary variable is uniformly distributed**

<table>
<thead>
<tr>
<th>L</th>
<th>AOSB</th>
<th>Total Variance [\sqrt{V(y_{SRSS})_{eq}}]</th>
<th>R.E. %</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.5069</td>
<td>0.75521</td>
<td>102.069</td>
</tr>
<tr>
<td>3</td>
<td>1.3601, 1.6833</td>
<td>0.75221</td>
<td>102.476</td>
</tr>
<tr>
<td>4</td>
<td>1.2673, 1.9914, 1.7621</td>
<td>0.75289</td>
<td>102.384</td>
</tr>
<tr>
<td>5</td>
<td>1.2618, 1.4585, 1.6247, 1.7959</td>
<td>0.75091</td>
<td>102.654</td>
</tr>
<tr>
<td>6</td>
<td>1.1785, 1.3848, 1.5199, 1.6805, 1.8476</td>
<td>0.75061</td>
<td>102.695</td>
</tr>
</tbody>
</table>

If the stratification variable follows the Right triangular distribution with pdf  
\[ f(x) = \frac{2(2-x)}{(b-a)^2}, \quad x \in [0,1] \], using \( a = 1 \) and \( b = 2 \), utilizing the  
cum\sqrt{k(x)} rule, we get the stratification points as given in Table VI.

**Table VI: AOSB and Variance when the auxiliary variable is Right-triangular distributed**

<table>
<thead>
<tr>
<th>L</th>
<th>AOSB</th>
<th>Total Variance [\sqrt{V(y_{SRSS})_{eq}}]</th>
<th>R.E. %</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.4387</td>
<td>0.67069</td>
<td>101.471</td>
</tr>
<tr>
<td>3</td>
<td>1.2623, 1.5808</td>
<td>0.66850</td>
<td>101.803</td>
</tr>
<tr>
<td>4</td>
<td>1.2001, 1.4086, 1.6593</td>
<td>0.66771</td>
<td>101.925</td>
</tr>
<tr>
<td>5</td>
<td>1.1583, 1.3234, 1.5071, 1.7153</td>
<td>0.66734</td>
<td>101.981</td>
</tr>
<tr>
<td>6</td>
<td>1.1209, 1.2601, 1.4044, 1.5645, 1.7473</td>
<td>0.66714</td>
<td>102.012</td>
</tr>
</tbody>
</table>

If the stratification variable follows the exponential distribution with pdf  
\[ f(x) = e^{-x} \quad x \in [0.5], \] utilizing the cum\sqrt{k(x)} rule, we get the stratification points as given in Table VII.

**Table VII: AOSB and Variance when the auxiliary variable is exponentially distributed**

<table>
<thead>
<tr>
<th>( \eta(x) = \lambda )</th>
<th>Total Variance [\sqrt{V(y_{SRSS})_{eq}}]</th>
<th>R.E. %</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.99659</td>
<td>0.23979</td>
<td>113.639</td>
</tr>
<tr>
<td>0.96859</td>
<td>0.23979</td>
<td>116.778</td>
</tr>
<tr>
<td>0.96157</td>
<td>0.23979</td>
<td>117.777</td>
</tr>
<tr>
<td>0.95418</td>
<td>0.23979</td>
<td>118.689</td>
</tr>
<tr>
<td>0.95133</td>
<td>0.23979</td>
<td>81.0268</td>
</tr>
</tbody>
</table>

If the stratification variable follows the standard normal distribution with pdf  
\[ f(x) = e^{-\frac{x^2}{2}}, \quad x \in [0,1] \], utilizing the  
cum\sqrt{k_3(x)} rule, we get the stratification points as given in Table VIII.

**Table VIII: AOSB and Variance when the auxiliary variable is Standard normally distributed**

<table>
<thead>
<tr>
<th>( \eta(x) = \lambda )</th>
<th>Total Variance [\sqrt{V(y_{SRSS})_{eq}}]</th>
<th>R.E. %</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.7018</td>
<td>0.08125</td>
<td>119.60</td>
</tr>
<tr>
<td>0.3577, 0.6815</td>
<td>0.07929</td>
<td>122.55</td>
</tr>
<tr>
<td>0.2675, 0.5191, 0.7547</td>
<td>0.07894</td>
<td>123.10</td>
</tr>
<tr>
<td>0.2296, 0.4259, 0.6235, 0.8088</td>
<td>0.07878</td>
<td>123.34</td>
</tr>
<tr>
<td>0.1982, 0.3652, 0.5252, 0.6806, 0.8389</td>
<td>0.07869</td>
<td>123.48</td>
</tr>
</tbody>
</table>

**CONCLUSION:**

For obtaining the stratification points under RSS, we have assumed different distributions for the auxiliary variable used as stratification variable. The proposed method of determining AOSB is used for all \( L = 1,2,\ldots,6 \) for \( L = 6 \) the efficiency shows decreasing trend. Although, the efficiency of cum\sqrt{k_3(x)} depends on the initial choice of
the number of classes but there is no theory which gives the best number of classes. Thus, the proposed method of $\text{cum} \sqrt[k]{k_3(x)}$ shows an increase in gain % in precision while selecting samples using RSS. This method works on a single auxiliary variable but in reality, surveys involve multiple auxiliary variables. Developing methods for multiple auxiliary variables as well as for other skewed distributions are possibilities for future work.

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REFERENCES