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A Flexible Circular Distribution with Applications to Biological, Geological, and Astronomical Data

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***Abstract*—In disciplines such as physics, biology and meteorology, where measurements like angles, directions and periodic phenomena are inherently cyclical, circular data is essential. Specialized techniques are needed to analyze such data in order to account for its wrap around character and guarantee insightful analysis. This encourages the development of innovative wrapped distribution and related models. In this research paper, we introduce the DUS-Wrapped Exponential (DWE) distribution, a new variation of the wrapped exponential distribution derived using the DUS transformation. Compared to the conventional wrapped distribution, the suggested distribution provides more flexibility. The hazard rate function, reliability, expectation, median, moments, and characteristic function are among the important distributional features that are obtained. The model parameter is estimated using the maximum likelihood estimation (MLE), Least Square (LS) and Weighted Least Square (WLS) method. To verify the reliability and consistency of the estimator, a simulation technique is employed. The practical applicability of the proposed DWE distribution is demonstrated through its application to three real-world circular datasets.**

***Index Terms*—Circular Data, DUS transformation, Maximum Likelihood Estimation, Monte Carlo, Wrapped Exponential Distribution.**

I. Introduction

In many scientific domains where measurements (angles or directions) are directed by nature, circular data emerge. Circular data are limited to a periodic scale, usually measured in degrees $[0^\circ, 360^\circ]$ or radians $[0, 2\pi]$, in contrast to linear data, which reside on the real line. Geology, which analyzes fault line directions, biology, which studies animal movement orientations, meteorology, which records wind directions, are among the fields that naturally provide such data. An artificial type of circular data emerges when periodic time data with a period L (such as daily or weekly cycles) are mapped onto a circle of circumference L and then rescaled to 2π, matching the circumference of the unit circle. For instance, time data recorded on a 24-hour clock can be transformed so that 00:00 corresponds to 0 and 24:00 to 2π. Applications cover a wide range of fields, including ecology, where it is used to investigate animal migration, movement, and orientation, such as turtle nesting routes. It aids in the analysis of ocean currents or wind directions in meteorology, and it is crucial for determining the locations of celestial objects in astronomy. In engineering, circular data is also essential for modeling cyclic phenomena, such as machine phase angles or time-of-day processes, which makes it essential for analyzing directed or periodic features. Another use for directional data is in the treatment of cancer; by precisely locating tumours, directional data helps to maximize radiation therapy.It assists in figuring out the exact orientations and angles for radiation delivery, reducing harm to healthy tissues. In order to improve treatment planning, directional data is also useful for researching how cancer cells travel throughout the body. One of the key characteristics of circular data is periodicity, meaning that values wrap around at a fixed boundary, making 0° and 360° (or 0 and 2π radians) equivalent. Another distinguishing feature is the lack of a natural origin, as the reference direction is often arbitrary. Additionally, the directional nature of circular data renders traditional statistical measures like the mean and variance inadequate, necessitating specialized methods such as the circular mean and circular variance.

Definition of circular distribution

For a univariate linear variable, the definition of a continuous distribution is analogous to that of a continuous circular variable, denoted by Θ. For a chosen initial direction and an orientation of the unit circle, the distribution of a random angle Θ, F, on the real line is defined by

$$F(x) = P(0 \leq \theta \leq x), \quad 0 \leq x \leq 2\pi$$

and

$$F(x + 2\pi) - F(x) = 1, \quad -\infty < x < \infty$$

A function f represents the probability density function of a distribution that is completely continuous if and only if

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1) $f(\theta) \geq 0$ almost everywhere on $(-\infty, \infty)$.
2) $f(\theta + 2\pi) = f(\theta)$ almost everywhere on $(-\infty, \infty)$.
3) $\int_{\theta_0}^{\theta_0+2\pi} f(\theta)\, d\theta = 1$.

Using the wrapping approach, a linear random variable X on R is wrapped around the unit circle to produce the circular variable Θ, which is literally $\theta = X \bmod 2\pi$. Therefore, the density has the expression

$$f(\theta) = \sum_{k=-\infty}^{\infty} g(\theta + 2\pi k)$$

where g(.) is the density function of X. In order to ensure periodicity and appropriateness for circular data analysis, this procedure entails mapping the linear domain onto the circular domain. Paul Levy, who invented wrapped distributions, was the first to propose this idea. Since then, this field has seen substantial advancements and a great deal of research. Jammalamadaka & Kozubowski (2004), for example, talked about circular distributions that are produced by encircling the circle with the classical exponential and Laplace distributions on the real line. Dattatreya Rao et al. (2007) wrapped popular life-testing models such as logistic, lognormal, Weibull, and extreme-value distributions to create new circular models. The wrapped weighted exponential distribution is a novel class of circular distributions created by Roy & Adnan (2012). In a different study, Roy & Adnan (2012) examined the wrapped generalized Gompertz distribution and talked about how ornithology might use it. By wrapping a geometric distribution, Jacob & Jayakumar (2013) recently developed a new family of circular distributions and examined its characteristics. Rao et al. (2013) talked about the wrapped gamma distribution's properties. Adnan & Roy (2014) developed wrapped variance gamma distribution. Subrahmanyam et al. (2017) introduced Wrapped Exponentiated Inverted Weibull Distribution and 76 turtles' orientation data after egg laying was analyzed using the wrapped exponential inverted Weibull distribution. Joshi & Jose (2018) developed Wrapped Lindley Distribution and Chesneau et al. (2021) introduced wrapped modified Lindley distribution and two real-world datasets were subjected to the wrapped modified Lindley distribution. 76 turtles' orientations for laying eggs are included in the first set, while 133 measurements of feldspar lath orientations in basalt are included in the second. For goodness of fit, this distribution was contrasted with the wrapped Lindley distribution, transmuted wrapped exponential distribution Yilmaz & Biçer (2018), and wrapped exponential distribution.

Transformation using the same parameters as the base distribution is easier to understand, more computationally efficient, and simpler to perform. It makes the model more robust, particularly when working with limited data, by avoiding the complexity and overfitting risk associated with adding new parameters. Additionally, this method guarantees quicker estimates convergence. Among these techniques is DUS transformation Kumar et al. (2015). The definition of the new distribution's pdf g(x) and cdf G(x) is as follows, assuming that f(x) and F(x) stand for the probability density function (pdf) and cumulative distribution function (cdf) of a baseline distribution, respectively.

$$g(x) = \frac{f(x) \cdot e^{F(x)}}{e - 1}$$

$$G(x) = \frac{e^{F(x)} - 1}{e - 1}$$

Studies conducted so far have demonstrated that DUS distributions derived from base distributions generally provide a better fit to datasets compared to original base distribution. Genç & Özbilen (2023) introduce D-UEHL distribution based on DUS transformation on unit exponentiated half-logistic distribution CDF and compare it with Weibull, Beta, Kumarswamy, UEHL. Irshad et al. (2021) introduced Generalized DUS Transformed Log Normal distribution and its application to Cancer and Heart Transplant Datasets and evaluated its performance by comparing it with Two parameter Log Normal distribution, the exponentiated Log Normal, Generalized half- normal, New- generalized Lindley distribution, modified Weibull distribution, Weibull Distribution. And the results demonstrated that DUS- transformed Log- Normal distribution provided a better fit, as evidence by lower AIC and BIC values. The main goal of this research is to create a more adaptable distribution for modelling circular data using the DUS transformation method. This distribution is called the DUS Wrapped Exponential (DWE). This study is noteworthy since it is the first time the DUS technique has been applied to a wrapped distribution. This paper is organised as follows: The DWE distribution's probability density function (PDF) and cumulative distribution function (CDF) are introduced in the section 2. Fundamental properties such as hazard rate function, stress-strength reliability, Trigonometric moments, the characteristic function, location and dispersion measures, the median, skewness, kurtosis, modality behaviour and order statistics are among the important characteristics of the TWE distribution that are also examined in section 3. The section 4 "Parameter Estimation Methods" uses Maximum Likelihood (ML), Least Square and Weighted Least Square techniques to solve the statistical estimate problem for the DWE distribution. The performance of the estimators is assessed by simulation experiments presented in the "Monte Carlo Simulation Study" section 5. Three real-world dataset is analyzed in the "Application to Real Data" section 6 to illustrate the process. In the final portion, the study comes to a close with a summary.

II. DWE Distribution

The wrapping method is a popular technique for obtaining circular distributions from standard families and is important for modelling circular data. The Wrapped Exponential (WE) distribution was introduced by Jammalamadaka and Kozubowski in this context. It is distinguished by its cumulative density function (CDF) and probability density function (PDF) as follows:

$$f(\theta) = \frac{\lambda e^{-\lambda\theta}}{1 - e^{-2\pi\lambda}} \tag{2.1}$$

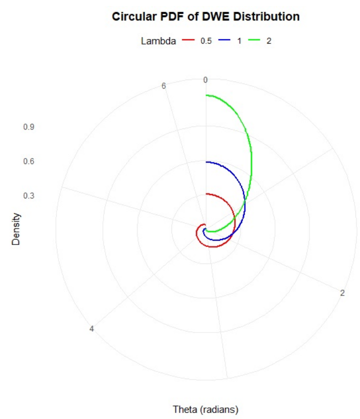

Fig. 1. PDF for DWE Distribution

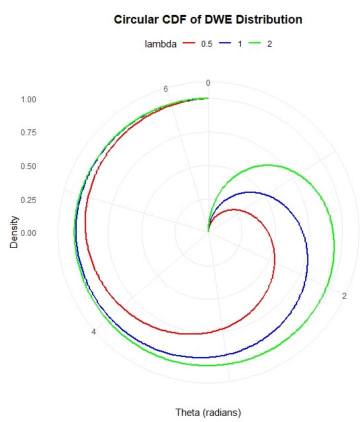

Fig. 2. CDF for DWE Distribution

and

$$F(\theta) = \frac{1 - e^{-\lambda\theta}}{1 - e^{-2\pi\lambda}} \tag{2.2}$$

where

$$\lambda > 0 \quad \text{and} \quad \theta \in [0, 2\pi)$$

The principal objective of this study is to propose a more flexible circular distribution than the classical Wrapped Exponential (WE) distribution, aimed at achieving better modelling of circular data. To achieve this we apply DUS method, utilizing expressions for equation (2.1) and (2.2) to derive pdf and cdf of a random variable Θ following DUS Wrapped Exponential distribution.

$$f(\theta) = \frac{\lambda e^{-\lambda\theta} \cdot e^{\frac{1-e^{-\lambda\theta}}{1-e^{-2\pi\lambda}}}}{(1 - e^{-2\pi\lambda})(e-1)} \tag{2.3}$$

and

$$F(\theta) = \frac{e^{\frac{1-e^{-\lambda\theta}}{1-e^{-2\pi\lambda}}} - 1}{e-1} \tag{2.4}$$

where

$$\lambda > 0 \quad \text{and} \quad \theta \in [0, 2\pi)$$

....

The pdf and cdf of DWE is shown in Figure 1 and Figure 2 using circular representations, respectively, for a range of parameter values. It is possible to see different models, peak intensities, and increasing-decreasing structures.

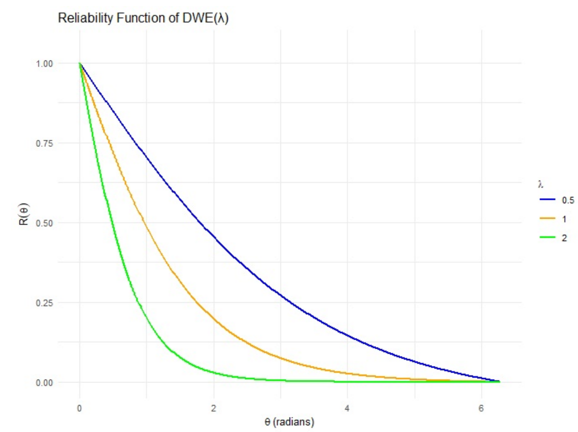

Fig. 3. PDF for DWE Distribution

III. Fundamental Properties

A. Reliability and Hazard function

The reliability function is defined as: 1- F(θ)
Hence, reliability function of DWE distribution is

$$R(\theta) = \frac{e - e^{\frac{1-e^{-\lambda\theta}}{1-e^{-2\pi\lambda}}}}{e-1}$$

This function represents the probability that the circular event has not occurred in the angular interval $[0, \theta)$. The exponential term in the numerator rises with increasing θ, causing R(θ) to monotonically drop from 1 to 0. This behaviour is in line with how reliability is interpreted: the system starts out with full reliability at $\theta = 0$, and as one moves around the circle, the likelihood of surviving that is, not witnessing the event decreases until it reaches zero at $\theta = 2\pi$.

Observations from the Plot of the Reliability Function:

- **Blue** ($\lambda = 0.5$): The curve slopes gently, suggesting that event occurrences are distributed more evenly within the circle.
- **Orange** ($\lambda = 1$): A moderate concentration of occurrences at lower angles is indicated by the curve's faster descent.
- **Green** ($\lambda = 2$): The function exhibits a sharp directional concentration of events at the circle's beginning (close to $\theta = 0$) as it declines sharply.

In linear distributions, the hazard rate function

$$hr(\theta) = \frac{f(\theta)}{1 - F(\theta)}$$

gives the **instantaneous failure rate** at time θ, assuming survival up to θ. For **circular distributions**, like our DWE(λ), the variable $\theta \in [0, 2\pi)$ represents *directions or angles*, not time. However, the hazard function still retains a similar interpretation, adapted for circular data:

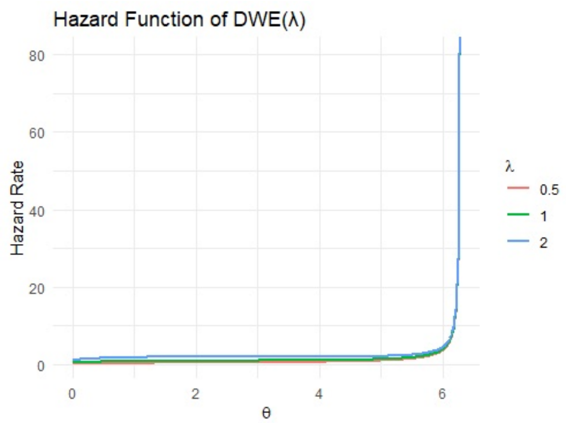

Fig. 4. Hazard for DWE Distribution

The Hazard Rate $hr(\theta)$ tells us how "likely" it is that an observation occurs around angle θ, given that it hasn't occurred in the interval $[0, \theta)$.

So although it doesn't mean "failure over time", it describes the **concentration** of the distribution in *angular terms*.
The hazard rate function hr of $\Theta \sim \text{DWE}(\lambda)$

$$hr(\theta) = \frac{f(\theta)}{1 - F(\theta)} = \frac{\lambda e^{-\lambda\theta} e^{\frac{1-e^{(-\lambda\theta)}}{1-e^{-2\pi\lambda}}}}{(1 - e^{-2\pi\lambda}) \left[e - e^{\frac{1-e^{(-\lambda\theta)}}{1-e^{2\pi\lambda}}} \right]}$$

where

$$\theta \in [0, 2\pi), \lambda > 0$$

Observations from the Plot of the Hazard Function:

- $\lambda = 0.5$: The distribution is more dispersed, the hazard rate is generally flat and increases slowly.
- $\lambda = 1$: A more distinct peak indicates greater focus in particular directions.
- $\lambda = 2$: A sharp spike early on the majority of observations are grouped at lower angles.

B. Characteristic Function

The DWE(λ) characteristic function is

$$\Phi_p = E(e^{ip\theta}) =$$

$$\frac{\lambda}{c(e-1)} \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \frac{1}{c^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \left[\frac{1 - e^{1-2\pi\lambda(1+k)}}{\lambda(1+k) - ip} \right] \quad (1)$$

Result:

$$[(a - ib)^{-r} = \left(a^2 + b^2\right)^{-\frac{r}{2}}] e^{i \arctan\left(\frac{b}{a}\right)} \quad \text{for } a, b \in R^+ \quad (2)$$

therefore

$$\{\lambda(1+k) - ip\}^{-1} = \left(\lambda^2(1+k)^2 + p^2\right)^{-1/2} e^{i \arctan\left(\frac{p}{\lambda(1+k)}\right)} \quad (3)$$

$$\Phi_p = \frac{\lambda}{c(e-1)} \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \frac{1}{c^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \left[\frac{1 - e^{1-2\pi\lambda(1+k)}}{\lambda^2(1+k)^2 + p^2} \right] \times e^{i \arctan\left(\frac{p}{\lambda(1+k)}\right)} \quad (4)$$

Now, since

$$\Phi_p = \rho_p \exp(i\mu_p) =$$

$$\frac{\lambda}{c(e-1)} \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \frac{1}{c^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \left[\frac{1 - e^{1-2\pi\lambda(1+k)}}{\lambda(1+k) - ip} \right] \quad (5)$$

Therefore,

$$\rho_p = \frac{\lambda}{c(e-1)} \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \frac{1}{c^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \left[\frac{1 - e^{1-2\pi\lambda(1+k)}}{\sqrt{\lambda^2(1+k)^2 + p^2}} \right] \quad (6)$$

and

$$\mu_p = e^{i \arctan\left(\frac{p}{\lambda(1+k)}\right)} \quad (7)$$

C. Trignometric moments and other properties

The pth trigonometric moment corresponds to the value of the characteristic function evaluated at p for a given circular random variable

$$\Theta \sim \text{DWE}(\lambda),$$

given p is an integer. The non-central trigonometric moments of the proposed distribution are therefore expressed as follows:

$$\alpha_p = \rho_p \cos(\mu_p)$$

and

$$\beta_p = \rho_p \sin(\mu_p),$$

according to definition of trignometric moments, which is

$$\phi_p = \alpha_p + i\beta_p$$

where $p = \pm 1, \pm 2, \ldots$ So,

$$\alpha_p = \frac{\lambda}{c(e-1)} \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \frac{1}{c^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \left[\frac{1 - e^{1-2\pi\lambda(1+k)}}{\sqrt{\lambda^2(1+k)^2 + p^2}} \right] \times \cos\left(\arctan \frac{p}{\lambda(1+k)} \right) \quad (8)$$

$$\beta_p = \frac{\lambda}{c(e-1)} \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \frac{1}{c^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \left[\frac{1 - e^{1-2\pi\lambda(1+k)}}{\sqrt{\lambda^2(1+k)^2 + p^2}} \right] \times \sin\left(\arctan \frac{p}{\lambda(1+k)} \right) \quad (9)$$

Now,

$$\bar{\alpha}_p = \rho_p \cos(\mu_p - p\mu_1)$$

and

$$\bar{\beta}_p = \rho_p \sin(\mu_p - p\mu_1),$$

yield the central trignometric moments. Therefore, DUS distribution's central trignometric moments are provided by

$$\bar{\alpha}_p = \frac{\lambda}{c(e-1)} \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \frac{1}{c^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \left[\frac{1 - e^{1-2\pi\lambda(1+k)}}{\sqrt{\lambda^2(1+k)^2 + p^2}} \right] \times \cos\left(\arctan \frac{p}{\lambda(1+k)} - p \arctan \frac{1}{\lambda(1+k)} \right) \quad (10)$$

$$\bar{\beta}_p = \frac{\lambda}{c(e-1)} \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \frac{1}{c^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \left[\frac{1 - e^{1-2\pi\lambda(1+k)}}{\sqrt{\lambda^2(1+k)^2 + p^2}} \right] \times \sin\left(\arctan \frac{p}{\lambda(1+k)} - p \arctan \frac{1}{\lambda(1+k)} \right) \tag{11}$$

Mean direction is given by μ_1 = arctan$\frac{1}{\lambda(1+k)}$
Angular Concentration = $\rho_1 =$

$$\frac{\lambda}{c(e-1)} \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \frac{1}{c^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \left[\frac{1 - e^{1-2\pi\lambda(1+k)}}{\sqrt{\lambda^2(1+k)^2 + 1}} \right] \tag{12}$$

Circular Variance, $V_0 = 1 - \rho =$

$$1 - \frac{\lambda}{c(e-1)} \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \frac{1}{c^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \left[\frac{1 - e^{-(2\pi\lambda(1+k)-1)}}{\sqrt{\lambda^2(1+k)^2 + 1}} \right] \tag{13}$$

Circular deviation, $\sigma_0 = \sqrt{-2log(1 - V_0)} =$

$$\sqrt{-2log \frac{\lambda}{c(e-1)} \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \frac{1}{c^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \left[\frac{1 - e^{1-2\pi\lambda(1+k)}}{\sqrt{\lambda^2(1+k)^2 + 1}} \right]} \tag{14}$$

Given a circular distribution, the median direction is a value M such that

$$\int_0^M f_\Theta(\theta)\, d\theta = \int_M^{2\pi} f_\Theta(\theta)\, d\theta = 0.5$$

The median of DWE(λ) distribution is

$$M = c \log \frac{e+1}{2} \tag{15}$$

The formula for the coefficient of skewness is

$$\xi_1^0 = \frac{\bar{\beta}_2}{V_0^{3/2}} =$$

$$\frac{\frac{\lambda}{c(e-1)} \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \frac{1}{c^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \left[\frac{1 - e^{-(2\pi\lambda(1+k)-1)}}{\sqrt{\lambda^2(1+k)^2 + 4}} \right]}{\left[1 - \frac{\lambda}{c(e-1)} \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \frac{1}{c^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \left[\frac{1 - e^{-(2\pi\lambda(1+k)-1)}}{\sqrt{\lambda^2(1+k)^2 + 1}} \right]\right]^{3/2}} \times \sin\left(\arctan\left(\frac{2}{\lambda(1+k)} \right) - 2 \arctan\left(\frac{1}{\lambda(1+k)} \right) \right) \tag{16}$$

Coefficient of kurtosis is given by

$$\xi_2^0 = \frac{\bar{\alpha}_2 - (1 - V_0)^4}{V_0^2} =$$

$$= \frac{\frac{\lambda}{c(e-1)} \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{c^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \times \left[\frac{1 - e^{1-2\pi\lambda(1+k)}}{\sqrt{\lambda^2(1+k)^2 + 4}} \right] \cos\left(\arctan\left(\frac{2}{\lambda(1+k)} \right) \right)}{D} - \frac{\left[\frac{\lambda}{c(e-1)} \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{c^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \left[\frac{1 - e^{1-2\pi\lambda(1+k)}}{\sqrt{\lambda^2(1+k)^2 + 1}} \right] \right]^4}{D} \tag{17}$$

$$D = 1 - \frac{\lambda}{c(e-1)} \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{c^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \left[\frac{1 - e^{1-2\pi\lambda(1+k)}}{\sqrt{\lambda^2(1+k)^2 + 1}} \right]^2 \tag{18}$$

D. Order Statistics

The probability density function (pdf) of ith order statistic Θ_i is denoted by

$$f_{\Theta_{(i)}}(\theta) = \frac{n!}{(i-1)!(n-i)!} F(\theta)^{i-1} f(\theta)(1 - F(\theta))^{n-i} =$$

$$\frac{n!}{(i-1)!(n-i)!(e-1)^n} \left[e^{\frac{1-e^{-\lambda\theta}}{c}} - 1 \right]^{i-1} \frac{\lambda e^{-\lambda\theta}}{c} e^{\frac{1-e^{-\lambda\theta}}{c}} \left[e - e^{\frac{1-e^{-\lambda\theta}}{c}} \right]^{n-i} \tag{19}$$

where

$$c = 1 - e^{-2\pi\lambda}$$

The probability density functions (PDFs) of the minimum and maximum order statistic from a sample of size n drawn from the DWE(λ) distribution (with $c = 1 - e^{-2\pi\lambda}$) are given by:

- **Minimum Order Statistic**:

$$f_{\Theta_{(1)}}(\theta) = \frac{n\lambda e^{-\lambda\theta} e^{\frac{1-e^{-\lambda\theta}}{c}}}{c(e-1)^2} \left[e - e^{\frac{1-e^{-\lambda\theta}}{c}} \right], \tag{20}$$

  This density is right-skewed, especially for larger values of λ, indicating that the first observation (i.e., the smallest angular value) tends to occur near $\theta = 0$. As λ increases, the distribution becomes more concentrated toward lower angular values.

- **Maximum Order Statistic**:

$$f_{\Theta_{(n)}}(\theta) = \frac{n\lambda e^{-\lambda\theta} e^{\frac{1-e^{-\lambda\theta}}{c}}}{c(e-1)^n} \left[e^{\frac{1-e^{-\lambda\theta}}{c}} - 1 \right]^{n-1}. \tag{21}$$

  This function is left-skewed and becomes more concentrated near 2π as n increases, meaning the last occurrence is likely to appear at higher angular values. The effect of increasing n results in a steeper tail towards $\theta = 2\pi$.

E. Entropy

Every probability distribution has some degree of uncertainty, as is widely known. An effective metric for quantifying this uncertainty is the entropy. If X is a random variable with probability density function f(x), then the Renyi entropy, as proposed by [Renyi (1961)], is defined as:

$$RE_\theta(\gamma) = \frac{\log[f^\gamma(\theta) d\theta]}{1 - \gamma}$$

where

$$\gamma > 0 \text{ and } \gamma \neq 1$$

$$\int_0^{2\pi} f^\gamma(\theta) d\theta = \left[\frac{\lambda}{(e-1)(1 - e^{-2\pi\lambda})} \right]^\gamma \int_0^{2\pi} \left[e^{-\lambda\theta} e^{1-e^{-\lambda\theta}/(1-e^{-2\pi\lambda})} \right]^\gamma \tag{22}$$

Shanon (1951) proposed the Shanon entropy, which is E[-logf(x)]. For $\gamma \longrightarrow 1$, it acts as a specific instance of Rennyi's entropy. Applying the L' Hospital Rule $\gamma \longrightarrow 1$ we obtain,

$$-\log f(x) =$$

$$\log(e-1) + \log(1 - e^{-2\pi\lambda}) - \log(\lambda) + \lambda\theta - \frac{1 - e^{-\lambda\theta}}{1 - e^{2\pi\lambda}} \tag{23}$$

Further, we will calculate E[-logf(x)] that gives shannon entropy in an open form.

IV. Parameter Estimation Methods

This section presents different methods for estimating the unknown parameter λ of the DUS–Wrapped Exponential distribution, namely the Maximum Likelihood Estimation (MLE), Least Squares (LS), and Weighted Least Squares (WLS) approaches.

A. Maximum Likelihood Estimation

This section outlines the maximum likelihood estimation (MLE) procedure for determining the model parameter λ. Consider $\theta_1, \theta_2, \ldots, \theta_n$ to be a random sample of size n from the DWE λ distribution. The corresponding log-likelihood function is given by:

$$\sum_{i=1}^{n} \left(\log(\lambda) - \lambda\theta_i + \frac{1 - e^{-\lambda\theta_i}}{1 - e^{-2\pi\lambda}} \right) \tag{24}$$

By setting the partial derivative of the log-likelihood function with respect to λ equal to zero, we obtain

$$\frac{\partial \log L}{\partial \lambda} =$$

$$\frac{n}{\lambda} - \sum_{i=1}^{n} \theta_i + \frac{\theta_i e^{-\lambda\theta_i} c - 2\pi e^{-2\pi\lambda}(1 - e^{-\lambda\theta_i})}{c^2} - \frac{n 2\pi e^{-2\pi\lambda}}{c} = 0 \tag{25}$$

To estimate λ, a numerical optimization technique is necessary as the likelihood equation does not provide a closed-form solution. In this work, the maxLik package in R is used to determine the maximum likelihood estimator (MLE) of λ, using the Newton-Raphson approach for iterative optimization.

B. Least Squares and Weighted Least Squares Estimation

Let Θ be a circular random variable following the DUS–Wrapped Exponential (DUS–WE) distribution with parameter $\lambda > 0$. Its cumulative distribution function (CDF) is given by

$$F(\theta) = \frac{e^{\frac{1 - e^{-\lambda\theta}}{1 - e^{-2\pi\lambda}}} - 1}{e - 1}, \tag{2.4}$$

where $0 \leq \theta < 2\pi$.

Let $\theta_{(1)} < \theta_{(2)} < \cdots < \theta_{(n)}$ denote the ordered random sample of size n drawn from the DUS–WE distribution.

The Least Squares (LS) method estimates λ by minimizing the squared deviations between the theoretical CDF and the empirical distribution function of the ordered observations. The LS objective function is defined as

$$Q_{LS}(\lambda) = \sum_{j=1}^{n} \left[F\big(\theta_{(j)}; \lambda\big) - \frac{j}{n+1} \right]^2. \tag{26}$$

Here, $F(\theta_{(j)}; \lambda)$ denotes the theoretical CDF evaluated at the jth ordered observation, while $\frac{j}{n+1}$ represents the expected value of the empirical distribution function (Swain et al., 1988). The LS estimator $\hat{\lambda}_{LS}$ is obtained by solving

$$\frac{\partial Q_{LS}(\lambda)}{\partial \lambda} = 0. \tag{27}$$

Although the LS estimator is simple and computationally convenient, it may exhibit bias, particularly for small or moderate sample sizes. To overcome this limitation, the Weighted Least Squares (WLS) method is employed, in which each squared deviation is multiplied by an appropriate weight. The WLS objective function is given by

$$Q_{WLS}(\lambda) = \sum_{j=1}^{n} w_j \left[F\big(\theta_{(j)}; \lambda\big) - \frac{j}{n+1} \right]^2, \tag{28}$$

where the weights w_j are defined as

$$w_j = \frac{(n+1)^2(n+2)}{j(n-j+1)}, \quad j = 1, 2, \ldots, n. \tag{29}$$

The WLS estimator $\hat{\lambda}_{WLS}$ is obtained by solving

$$\frac{\partial Q_{WLS}(\lambda)}{\partial \lambda} = 0. \tag{30}$$

Compared to the LS estimator, the WLS approach generally yields less biased and more efficient estimates, especially for moderate sample sizes.

V. Simulation Study

To evaluate the performance of the estimators for the parameter λ, a Monte Carlo simulation study is conducted. For different values of λ, random samples of sizes 25, 50, 100, and 200 are generated, and the process is carried out $N = 1000$ times. The following describes the simulation algorithm, and Table 1 provides a summary of the results:

Step 1: Use the uniform distribution $U(0, 1)$ to generate a random number u.

Step 2: Equate the cumulative distribution function (CDF), as specified in Equation (2.4), to u and solve for θ to obtain a circular random variable from the DUS Wrapped Exponential distribution.

Step 3: Substitute the generated random sample into the estimating equations corresponding to the different estimation methods, namely the Maximum Likelihood Estimation (MLE), Least Squares (LS), and Weighted Least Squares (WLS). The MLE of λ is obtained using the `maxLik` package in R, which employs the Newton Raphson optimisation method, while the LS and WLS estimates are computed by minimizing their respective objective functions.

Step 4: Repeat Steps 1–3 independently N times. The mean squared error (MSE) and average bias of $\hat{\lambda}$ are then calculated using the following formulas, where λ denotes the true value of the parameter:

$$\text{Bias}(\hat{\lambda}) = \frac{1}{N} \sum_{i=1}^{N} (\hat{\lambda}_i - \lambda), \qquad \text{MSE}(\hat{\lambda}) = \frac{1}{N} \sum_{i=1}^{N} (\hat{\lambda}_i - \lambda)^2.$$

The simulation results show that bias and MSE are consistent and decrease with increasing sample size. For most parameter choices, the WLS estimator consistently yields the lowest MSE among the three methods, even though the MLE becomes competitive for larger samples. The LS estimator has a slightly higher MSE and a lower efficiency despite its simplicity.

TABLE I
BIAS AND MSE OF ESTIMATORS OF λ FOR THE DUS WRAPPED EXPONENTIAL DISTRIBUTION

Method	n	$\hat{\lambda}$	
		Bias	MSE
$\lambda = 1.0$			
LS	30	0.0164	0.0376
	50	0.0111	0.0224
	100	0.0110	0.0114
	200	0.0028	0.0053
WLS	30	0.0148	0.0348
	50	0.0100	0.0208
	100	0.0111	0.0104
	200	0.0026	0.0048
MLE	30	0.0240	0.0302
	50	0.0142	0.0181
	100	0.0130	0.0090
	200	0.0028	0.0040
$\lambda = 1.5$			
LS	30	0.0323	0.0826
	50	0.0190	0.0466
	100	0.0156	0.0234
	200	0.0049	0.0120
WLS	30	0.0308	0.0753
	50	0.0170	0.0422
	100	0.0158	0.0211
	200	0.0049	0.0107
MLE	30	0.0454	0.0673
	50	0.0225	0.0357
	100	0.0203	0.0183
	200	0.0071	0.0087
$\lambda = 2.0$			
LS	30	0.0338	0.1467
	50	0.0232	0.0858
	100	0.0076	0.0424
	200	0.0112	0.0192
WLS	30	0.0302	0.1331
	50	0.0213	0.0791
	100	0.0080	0.0387
	200	0.0121	0.0177
MLE	30	0.0517	0.1132
	50	0.0308	0.0671
	100	0.0158	0.0340
	200	0.0169	0.0156

VI. Real-World Data Analysis

In this section, we assess the performance of the proposed DWE model by comparing it with several competing circular distributions, using three real-world datasets consisting of directional observations measured in degrees around the unit circle. These data sets are transformed into floor radians using the formula π* data/180 then we determine MLE and conduct the analysis. It should be noted that maxLik package is used to calculate MLEs.

A. Data set 1

The first set of data is about measurements of long-axis orientation of 133 feldspar laths in basalt.

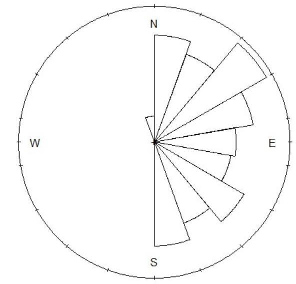

Fig. 5. rose plot for data set 1

Source: Smith (1988, set 28-6-lco.prn); data kindly supplied by Ms Nicola Smith.
176, 66, 18, 53, 50, 51, 39, 131, 86, 144, 54, 104, 90, 132, 162, 104, 39, 176, 6, 144, 134, 50, 37, 138, 121, 39, 143, 12, 49, 97, 140, 72, 59, 148, 68, 93, 132, 10, 1, 133, 13, 1, 174, 58, 63, 170, 140, 44, 57, 160, 83, 45, 39, 36, 50, 174, 121, 55, 113, 54, 60, 129, 127, 163, 137, 111, 63, 109, 49, 5, 170, 56, 64, 98, 68, 124, 58, 11, 153, 4, 12, 54, 178, 169, 87, 56, 86, 132, 65, 144, 145, 13, 21, 170, 63, 3, 37, 161, 38, 145, 82, 108, 29, 103, 5, 51, 5, 59, 168, 152, 164, 61, 38, 54, 52, 80, 69, 5, 30, 14, 61, 0, 73, 21, 143, 168, 119, 61, 172, 124, 107, 52, 91

TABLE II
LOG-LIKELIHOOD AND INFORMATION CRITERIA OF DATA SET 1.

Model	- log-likelihood	AIC	BIC
DWE	176.147	354.2914	357.1844
TWE	175.2225	354.4449	360.2256
WL	176.655	355.3121	358.2024
WE	182.1457	366.2914	369.1818

TABLE III
SUMMARY STATISTICS FOR THE CIRCULAR DATASET 1

Statistic	**Value**
$\hat{\lambda}$ (lambda)	0.840117
$\bar{x}$ (Mean Cosine)	0.0848969
$\bar{y}$ (Mean Sine)	0.620539
Resultant Length ($\bar{R}$)	0.626319
Mean Direction (radians)	1.43483
Mean Direction (degrees)	82.2096
Circular Variance	0.373681
Circular Std Deviation	0.967362
Circular Dispersion	0.59663
Circular Skewness	-0.118564
Circular Kurtosis	-0.00523029

Four wrapped distributions were fitted to the dataset — the DUS Wrapped Exponential (DWE), the Transmuted Wrapped Exponential (TWE), the Wrapped Lindley (WL), and the Wrapped Exponential (WE). Each model's parameters were estimated, and then the log-likelihood, Akaike Information Criterion (AIC), and Bayesian Information Criterion (BIC)

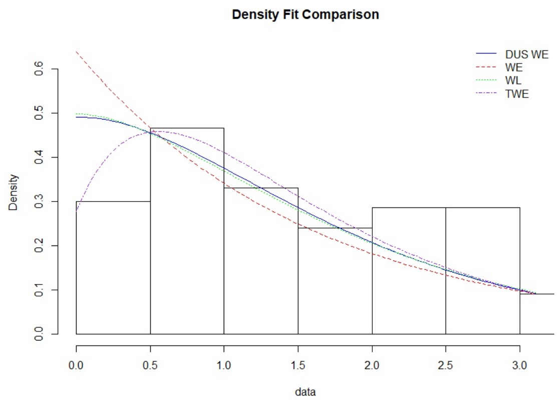

Fig. 6. density fit comparison of data set 1

were calculated as model selection criteria. In general, a model with lower AIC and BIC values is favoured since it shows a better balance between complexity and model fit. The DWE distribution outperforms the TWE, WL, and WE distributions among the four models taken into consideration, displaying the lowest AIC (354.2914) and BIC (357.1844) values. This indicates unequivocally how well the DWE distribution captures the data's fundamental structure.

B. Data set 2

Declination and right ascension are the coordinate systems used in the second data set, which measures the arrival directions of low cosmic ray showers.
The dataset was reported by Fisher et al. (1993) and consist of following observation in degree:
315, 198, 99, 50, 86, 221, 0, 14, 63, 176, 185, 207, 221, 230, 243, 347, 342, 311, 293, 284, 216, 207, 144, 149, 77, 41, 27, 14, 5, 41, 36, 50, 63, 77, 77, 113, 153, 176, 203, 216, 252, 279, 288, 311, 320, 342, 356, 347, 347, 342, 338, 329, 293, 284, 261, 252, 257, 261, 261, 252, 230, 230, 216, 212, 207, 194, 158, 117, 99, 95, 14, 18, 27, 23, 50, 63, 77, 167, 176, 185, 194, 221, 216, 230, 234, 248, 279, 297, 324, 234, 185, 162, 144, 144, 108, 104, 86, 77, 68, 63, 27, 95, 99, 122, 140, 144, 171, 176, 185, 198, 216, 216, 234, 324, 311, 293, 275, 266, 207, 203, 212, 198, 140, 117, 117, 86, 68, 32, 14, 59, 68, 68, 72, 86, 104, 212, 153, 216, 342, 338, 9, 230, 221, 203, 198, 99, 86, 68, 45

TABLE IV
LOG-LIKELIHOOD AND INFORMATION CRITERIA OF DATA SET 2.

Model	log-likelihood	AIC	BIC
DWE	272.9777	547.9553	550.9553
TWE	272.7574	549.5148	555.5227
WL	273.0644	548.1288	551.1328
WE	273.5696	549.1392	552.1432

The DWE model performs the best in terms of model fit and parsimony among these, producing the lowest AIC (547.9553) and BIC (550.9553) as mentioned in table 4. The DWE's advantage in both AIC and BIC indicates its higher efficiency

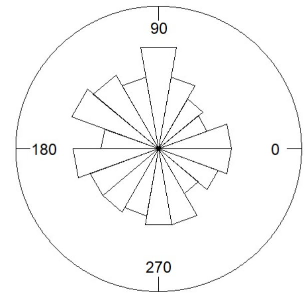

Fig. 7. rose plot for data set 2

TABLE V
SUMMARY STATISTICS FOR THE CIRCULAR DATASET 2

Statistic	Value
$\hat{\lambda}$ (lambda)	0.194253
$\bar{x}$ (Mean Cosine)	-0.0830776
$\bar{y}$ (Mean Sine)	0.00977824
Resultant Length ($\bar{R}$)	0.0836511
Mean Direction (radians)	3.02443
Mean Direction (degrees)	173.287
Circular Variance	0.916349
Circular Std Deviation	2.2276
Circular Dispersion	10.9544
Circular Skewness	0.158077
Circular Kurtosis	-0.0301347

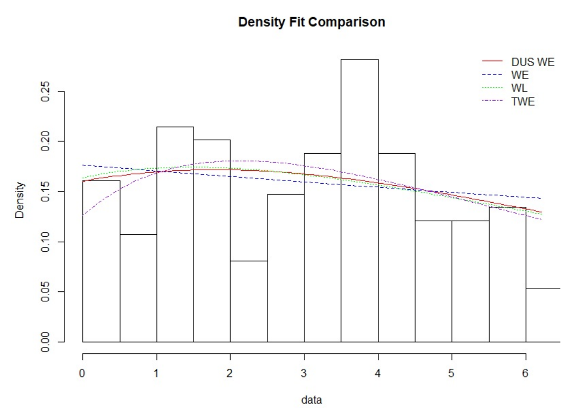

Fig. 8. density fit comparsion of data set 2

in capturing the underlying structure of the data without needless complication, even if the log-likelihood values of all models are rather close. This proves that of the four models taken into consideration, the DWE is the most suitable and reliable.

C. Data set 3

The third data set is of Sun compass orientations of 50 star-head topminows, measured under heavily overcast conditions. source: Goodyear (1970, Figure 1D)
The following observations, in degrees, are included in this dataset:

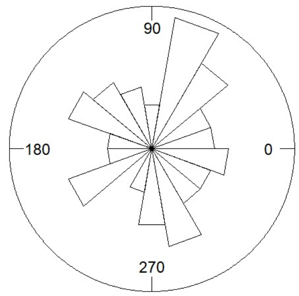

Fig. 9. rose plot for data set 3

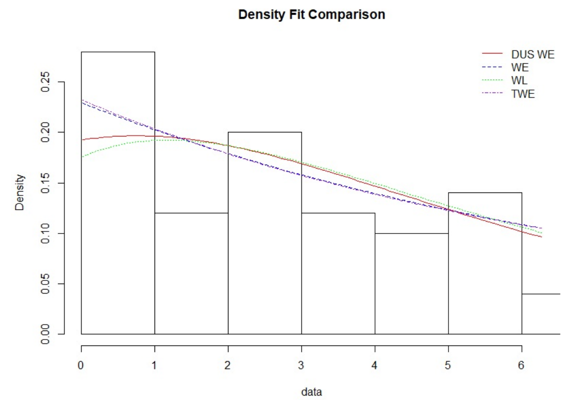

Fig. 10. density fit comparison of data set 3

2, 9, 18, 24, 30, 35, 35, 39, 39, 44, 44, 49, 56, 70, 76, 76, 81, 86, 01, 112, 121, 127, 133, 134, 138, 147, 152, 157, 166, 171, 177, 187, 206, 210, 211, 215, 238, 246, 269, 270, 285, 292, 305, 315, 325, 328, 329, 343, 354, 359

TABLE VI
LOG-LIKELIHOOD AND INFORMATION CRITERIA OF DATA SET 3.

Model	log-likelihood	AIC	BIC
DWE	91.20098	184.402	186.314
TWE	90.61005	185.22	189.044
WL	94.81907	191.638	193.550
WE	95.8932	193.516	195.178

TABLE VII
SUMMARY STATISTICS FOR THE CIRCULAR DATASET 3

Statistic	Value
$\hat{\lambda}$ (lambda)	0.2701172150410228
$\bar{x}$ (Mean Cosine)	0.10814992356604383
$\bar{y}$ (Mean Sine)	0.1372945545692598
Resultant Length ($\bar{R}$)	0.1747747140799051
Mean Direction (radians)	0.9035870157407518
Mean Direction (degrees)	51.77172242476617
Circular Variance	0.8252252859200949
Circular Std Deviation	1.8677566664412588
Circular Dispersion	4.721651471521279
Circular Skewness	-0.1282530086474511
Circular Kurtosis	0.015492714562468155

The DWE model yields the lowest AIC (184.402) and BIC (186.314) in table 6, indicating the best balance between model fit and simplicity. Despite similar log-likelihood values, the DWE outperforms the other models in both AIC and BIC, confirming it as the most efficient and robust choice for this data.Thomas et al. (2021)

VII. CONCLUSION

By applying the DUS transformation on the Wrapped Exponential (WE) distribution, we provide a new circular distribution in this paper called the DUS Wrapped Exponential (DWE). We provide a thorough analysis of the DWE's behavioural characteristics by deriving its probability density function (PDF) and cumulative distribution function (CDF). A thorough grasp of the DWE's mathematical structure is provided by the rigorous determination of its mean, characteristic function, trigonometric moments, and other essential characteristics. We derive the Maximum Likelihood Estimators (MLEs), the Least Squares and the Weighted Least Squares to estimate the unknown parameters of the DWE and evaluate their performance using a Monte Carlo simulation analysis. The simulation results illustrate the effectiveness and resilience of the estimating technique by showing that the bias and mean squared error (MSE) of the estimators both decrease with increasing sample size. Lastly, we compare the modelling performance of the DWE distribution with that of the Transmuted Wrapped Exponential (TWE), Wrapped Lindley (WL), and Wrapped Exponential (WE) distributions using three real-world datasets. We find that the DWE distribution consistently performs better than the competing models using likelihood-based metrics like AIC and BIC, making it the most appropriate and economical model for the datasets in question.

VIII. DECLARATIONS

Ethics approval and consent to participate. Not applicable

Consent for publication. Authors read and approved the final version of this article

Availability of data and materials. All data sources are given in the paper with its related references.

Competing interests. The authors declare that they have no competing interests to report regarding the present study.

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Authors' contributions. Author PS prepared the manuscript, including figures and tables. Author SKS verified the computations, edited and reviewed the final manuscript.

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